

# Capacitary estimates of solutions of semilinear parabolic equations

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## Abstract

We prove that any positive solution of  $\partial_t u - \Delta u + u^q = 0$  ( $q > 1$ ) in  $\mathbb{R}^N \times (0, \infty)$  with initial trace  $(F, 0)$ , where  $F$  is a closed subset of  $\mathbb{R}^N$  can be represented, up to two universal multiplicative constants, by a series involving the Bessel capacity  $C_{2/q, q'}$ . As a consequence we prove that there exists a unique positive solution of the equation with such an initial trace. We also characterize the blow-up set of  $u(x, t)$  when  $t \downarrow 0$ , by using the "density" of  $F$  expressed in terms of the  $C_{2/q, q'}$ -Bessel capacity.

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*Key words.* Heat equation; singularities; Borel measures; Besov spaces; real interpolation; Bessel capacities; quasi-additivity; capacitary measures; Wiener type test; initial trace.

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# 1 Introduction

Let  $T \in (0, \infty]$  and  $Q_T = \mathbb{R}^N \times (0, T]$  ( $N \geq 1$ ). If  $q > 1$  and  $u \in C^2(Q_T)$  is nonnegative and verifies

$$\partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_T, \quad (1.1)$$

it has been proven by Marcus and Véron [28] that there exists a unique outer-regular positive Borel measure  $\nu$  in  $\mathbb{R}^N$  such that

$$\lim_{t \rightarrow 0} u(., t) = \nu, \quad (1.2)$$

in the sense of Borel measures; the set of such measures is denoted by  $\mathfrak{B}_+^{reg}(\mathbb{R}^N)$ . To each of its element  $\nu$  is associated a unique couple  $(\mathcal{S}_\nu, \mu_\nu)$  (we write  $\nu \approx (\mathcal{S}_\nu, \mu_\nu)$ ) where  $\mathcal{S}_\nu$ , *the singular part* of  $\nu$ , is a closed subset of  $\mathbb{R}^N$  and  $\mu_\nu$ , *the regular part* is a nonnegative Radon measure on  $\mathcal{R}_\nu = \mathbb{R}^N \setminus \mathcal{S}_\nu$ . In this setting, relation (1.2) has the following meaning :

$$\begin{aligned} (i) \quad & \lim_{t \rightarrow 0} \int_{\mathcal{R}_\nu} u(., t) \zeta dx = \int_{\mathcal{R}_\nu} \zeta d\mu_\nu, & \forall \zeta \in C_0(\mathcal{R}_\nu), \\ (ii) \quad & \lim_{t \rightarrow 0} \int_{\mathcal{O}} u(., t) dx = \infty, & \forall \mathcal{O} \subset \mathbb{R}^N \text{ open, } \mathcal{O} \cap \mathcal{S}_\nu \neq \emptyset. \end{aligned} \quad (1.3)$$

The measure  $\nu$  is by definition *the initial trace* of  $u$  and denoted by  $Tr_{\mathbb{R}^N}(u)$ . It is wellknown that equation (1.1) admits a critical exponent

$$1 < q < q_c = 1 + \frac{N}{2}.$$

This is due to the fact, proven by Brezis and Friedman [7], that if  $q \geq q_c$ , isolated singularities of solutions of (1.1) in  $\mathbb{R}^N \setminus \{0\}$  are removable. Conversely, if  $1 < q < q_c$ , it is proven by the same authors that for any  $k > 0$ , equation (1.1) admits a unique solution  $u_{k\delta_0}$  with initial data  $k\delta_0$ . This existence and uniqueness results extends in a simple way if the initial data  $k\delta_0$  is replaced by any Radon measure  $\mu$  in  $\mathbb{R}^N$  (see [6]). Furthermore, if  $k \rightarrow \infty$ ,  $u_{k\delta_0}$  increases and converges to a positive, radial and self-similar solution  $u_\infty$  of (1.1). Writing it under the form  $u_\infty(x, t) = t^{-\frac{1}{q-1}} f(|x|/\sqrt{t})$ ,  $f$  is a positive solution of

$$\begin{cases} \Delta f + \frac{1}{2}y \cdot Df + \frac{1}{q-1}f - f^q = 0 & \text{in } \mathbb{R}^N \\ \lim_{|y| \rightarrow \infty} |y|^{\frac{2}{q-1}} f(y) = 0. \end{cases} \quad (1.4)$$

The existence, uniqueness and the expression of the asymptotics of  $f$  has been studied thoroughly by Brezis, Peletier and Terman in [8]. Later on, Marcus and Véron proved in [28] that in the same range of exponents, for any  $\nu \in \mathfrak{B}_+^{reg}(\mathbb{R}^N)$ , the Cauchy problem

$$\begin{cases} \partial_t u - \Delta u + u^q = 0 & \text{in } Q_\infty, \\ Tr_{\mathbb{R}^N}(u) = \nu, \end{cases} \quad (1.5)$$

admits a unique positive solution. This result means that the initial trace establishes a one to one correspondence between the set of positive solutions of (1.1) and  $\mathfrak{B}_+^{reg}(\mathbb{R}^N)$ . A key step for proving the uniqueness is the following inequalities

$$t^{-\frac{1}{q-1}} f(|x-a|/\sqrt{t}) \leq u(x, t) \leq ((q-1)t)^{-\frac{1}{q-1}} \quad \forall (x, t) \in Q_\infty, \quad (1.6)$$

valid for any  $a \in \mathcal{S}_\nu$ . As a consequence of Brezis and Friedman's result, if  $q \geq q_c$ , i.e. in the *supercritical range*, Problem (1.5) may admit no solution at all. If  $\nu \in \mathfrak{B}_+^{reg}(\mathbb{R}^N)$ ,  $\nu \approx (\mathcal{S}_\nu, \mu_\nu)$ , the necessary and sufficient conditions for the existence of a maximal solution  $u = \bar{u}_\nu$  to Problem (1.5) are obtained in [28] and expressed in terms of the the Bessel capacity  $C_{2/q, q'}$ , (with  $q' = q/(q-1)$ ). Furthermore, uniqueness does not hold in general as it was pointed out by Le Gall [23]. In the particular case where  $\mathcal{S}_\nu = \emptyset$  and  $\nu$  is simply the Radon measure  $\mu_\nu$ , the necessary and sufficient condition for solvability is that  $\mu_\nu$  does not charge Borel subsets with  $C_{2/q, q'}$ -capacity zero. This result was already proven by Baras and Pierre [5] in the particular case of bounded measures and extended by Marcus and Véron [28] to the general case. We denote by  $\mathfrak{M}_+^q(\mathbb{R}^N)$  the positive cone of the space  $\mathfrak{M}^q(\mathbb{R}^N)$  of Radon measures which do not charge Borel subsets with zero  $C_{2/q, q'}$ -capacity. Notice that  $W^{-2/q, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$  is a subset of  $\mathfrak{M}_+^q(\mathbb{R}^N)$  where  $\mathfrak{M}_+^b(\mathbb{R}^N)$  is the cone of positive bounded Radon measures in  $\mathbb{R}^N$ . For such measures, uniqueness always holds and we denote  $\bar{u}_{\mu_\nu} = u_{\mu_\nu}$ .

In view of the already known results concerning the parabolic equation, it is useful to recall the main advanced results previously obtained for the stationary equation

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \quad (1.7)$$

in a smooth bounded domain  $\Omega$  of  $\mathbb{R}^N$ . This equation has been intensively studied since 1993, both by probabilists (Le Gall, Dynkin, Kuznetsov) and by analysts (Marcus, Véron). The existence of a *boundary trace* for positive solutions, in the class of outer-regular positive Borel measures on  $\partial\Omega$ , is proven by Le Gall [22], [23] in the case  $q = N = 2$ , by probabilistic methods, and by Marcus and Véron in [26], [27] in the general case  $q > 1$ ,  $N > 1$ . The existence of a critical exponent  $q_e = (N+1)/(N-1)$  is due to Gmira and Véron [14] who shew that, if  $q \geq q_e$  boundary isolated singularities of solutions of (1.7) are removable, which is not the case if  $1 < q < q_e$ . In this *subcritical case* Le Gall and Marcus and Véron proved that the boundary trace establishes a one to one correspondence between positive solutions of (1.7) in  $\Omega$  and outer regular positive Borel measures on  $\partial\Omega$ . This fundamental result does not hold in the *supercritical case*  $q \geq q_e$ . In [12] Dynkin and Kuznetsov introduced the notion of  $\sigma$ -moderate solution which means that  $u$  is a positive solution of (1.7) such that there exists an increasing sequence of positive Radon measures on  $\partial\Omega$   $\{\mu_n\}$  belonging to  $W^{-2/q, q'}(\partial\Omega)$  such that the corresponding solutions  $v = v_{\mu_n}$  of

$$\begin{cases} -\Delta v + v^q = 0 & \text{in } \Omega \\ v = \mu_n & \text{in } \partial\Omega \end{cases} \quad (1.8)$$

converges to  $u$  locally uniformly in  $\Omega$ . This class of solutions plays a fundamental role since Dynkin and Kuznetsov proved that a  $\sigma$ -moderate solution of (1.7) is uniquely determined by its *fine trace*, a new notion of trace introduced in order to avoid the non-uniqueness phenomena. Later on, it is proved by Mselati (if  $q = 2$ ) [36], then by Dynkin (if  $q_e \leq q \leq 2$ ) [10] and finally by Marcus with no restriction on  $q$  [25], that all the positive solutions of (1.7) are  $\sigma$ -moderate. One of the key-stones element in their proof (partially probabilistic) is the fact that the maximal solution  $\bar{u}_K$  of (1.7) with a boundary trace vanishing outside a compact subset  $K \subset \partial\Omega$  is indeed  $\sigma$ -moderate. This deep result was obtained by a combination of probabilistic and analytic methods by Mselati [36] in the case  $q = 2$  and by purely analytic tools by Marcus and Véron [31], [32] in the case  $q \geq q_e$ . Defining  $\underline{u}_K$  as the largest  $\sigma$ -moderate solution of

(1.7) with a boundary trace concentrated on  $K$ , the crucial step in Marcus-Véron's proof (non probabilistic) is the bilateral estimate satisfied by  $\bar{u}_K$  and  $\underline{u}_K$

$$C^{-1}\rho(x)W_K(x) \leq \underline{u}_K(x) \leq \bar{u}_K(x) \leq C\rho(x)W_K(x). \quad (1.9)$$

In this expression  $C = C(\Omega, q)$ ,  $\rho(x) = \text{dist}(x, \partial\Omega)$  and  $W_F(x)$  is the *elliptic capacitary potential* of  $K$  defined by

$$W_K(x) = \sum_{-\infty}^{\infty} 2^{-\frac{m(q+1)}{q-1}} C_{2/q, q'}(2^m K_m(x)), \quad (1.10)$$

where  $K_m(x) = K \cap \{z : 2^{-m-1} \leq |z - x| \leq 2^{-m}\}$ , the Bessel capacity being relative to  $\mathbb{R}^{N-1}$ . Note that, using a technique introduced in [27], inequality  $\bar{u}_K \leq C^2 \underline{u}_K$  implies  $\underline{u}_K = \bar{u}_K$ .

The aim of this article is to initiate the fine study of the complete initial trace problem for positive solutions of (1.1) in the supercritical case  $q \geq q_c$  and to give in particular the parabolic counterparts of the results of [36], [31] and [32]. Extending Dynkin's ideas to the parabolic case, we introduce the following notion

**Definition 1.1** *A positive solution  $u$  of (1.1) is called  $\sigma$ -moderate if there exists an increasing sequence  $\{\mu_n\} \subset W^{-2/q, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$  such that the corresponding solution  $u := u_{\mu_n}$  of*

$$\begin{cases} \partial_t u - \Delta u + u^q = 0 & \text{in } Q_\infty \\ u(x, 0) = \mu_n & \text{in } \mathbb{R}^N, \end{cases} \quad (1.11)$$

*converges to  $u$  locally uniformly in  $Q_\infty$ .*

If  $F$  is a closed subset of  $\mathbb{R}^N$ , we denote by  $\bar{u}_F$  the maximal solution of (1.1) with an initial trace vanishing on  $F^c$ , and by  $\underline{u}_F$  the maximal  $\sigma$ -moderate solution of (1.1) with an initial trace vanishing on  $F^c$ . Thus  $\underline{u}_F$  is defined by

$$\underline{u}_F = \sup\{u_\mu : \mu \in W^{-2/q, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N), \mu(F^c) = 0\}, \quad (1.12)$$

(and clearly  $W^{-2/q, q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$  can be replaced by  $\mathfrak{M}_+^q(\mathbb{R}^N)$ ). One of the main goal of this article is to prove that  $\bar{u}_F$  is  $\sigma$ -moderate and more precisely,

**Theorem 1.2** *For any  $q > 1$  and any closed subset  $F$  of  $\mathbb{R}^N$ ,  $\bar{u}_F = \underline{u}_F$ .*

We define below a set function which will play a fundamental role in the sequel.

**Definition 1.3** *Let  $F$  be a closed subset of  $\mathbb{R}^N$ . The Bessel parabolic capacitary potential  $W_F$  of  $F$  is defined by*

$$W_F(x, t) = \frac{1}{t^{\frac{N}{2}}} \sum_{n=0}^{\infty} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q, q'}\left(\frac{F_n}{d_{n+1}}\right) \quad \forall (x, t) \in Q_\infty, \quad (1.13)$$

where  $C_{2/q, q'}$  is the  $N$ -dimensional Bessel capacity,  $d_n = \sqrt{nt}$  and  $F_n = \{y \in F : d_n \leq |x - y| \leq d_{n+1}\}$ .

In our study, it is useful to introduce a variant of  $W_F$  with the help of the Besov capacity: if  $\Omega \subset \mathbb{R}^N$  is a bounded domain, we set

$$\|\phi\|_{B_{2/q,q'}} = \left( \iint_{\Omega \times \Omega} \frac{|\phi(x) - \phi(y)|^{q'}}{|x - y|^{N + \frac{2}{q-1}}} dx dy \right)^{1/q'}, \quad (1.14)$$

if  $1 < 2/q < 1$ , and  $\|\phi\|_{B_{1,2}} = \|\nabla \phi\|_{L^2}$  if  $2/q = 1$  (i.e.  $N = 2$  and  $q = 2$ ). The Besov capacity of a compact set  $K \subset \Omega$  relative to  $\Omega$  is expressed by

$$R_{2/q,q'}^\Omega = \inf \left\{ \|\phi\|_{B_{2/q,q'}}^{q'} : \phi \in C_0^\infty(\Omega), 0 \leq \phi \leq 1, \eta = 1 \text{ on } K \right\}. \quad (1.15)$$

The Besov-parabolic capacity potential  $\tilde{W}_F$  of  $F$  is defined by

$$\tilde{W}_F(x, t) = t^{-\frac{N}{2}} \sum_{n=0}^{\infty} d_{n+1}^{N - \frac{2}{q-1}} e^{-\frac{n}{4}} R_{2/q,q'}^{\Gamma_n} \left( \frac{F_n}{d_{n+1}} \right) \quad \forall (x, t) \in Q_\infty, \quad (1.16)$$

where  $\Gamma_n = B_{d_{n+1}} \setminus \overline{B_{d_n}}$ . The Besov-parabolic capacity potential is equivariant with respect to the same scaling transformation which let (1.1) invariant in the sense that, for any  $\ell > 0$ ,

$$\ell^{\frac{1}{q-1}} \tilde{W}_F(\sqrt{\ell}x, \ell t) = \tilde{W}_F(x, t) \quad \forall (x, t) \in Q_\infty. \quad (1.17)$$

and we prove that there exists  $c = c(N, q) > 0$  such that

$$c^{-1} \tilde{W}_F(x, t) \leq W_F(x, t) \leq c \tilde{W}_F(x, t) \quad \forall (x, t) \in Q_\infty. \quad (1.18)$$

One of the tool for proving Theorem 1.2 is the following bilateral estimate which is only meaningful in the supercritical case, otherwise it reduces to (1.6);

**Theorem 1.4** *For any  $q \geq q_c$  there exist two positive constants  $C_1 \geq C_2 > 0$ , depending only on  $N$  and  $q$  such that for any closed subset  $F$  of  $\mathbb{R}^N$ , there holds*

$$C_2 W_F(x, t) \leq \underline{u}_F(x, t) \leq \overline{u}_F(x, t) \leq C_1 W_F(x, t) \quad \forall (x, t) \in Q_\infty. \quad (1.19)$$

Actually our result is more general since the upper estimate in (1.19) is valid for *any* positive solution of

$$\partial_t u - \Delta u + u^q \leq 0 \quad \text{in } Q_T \quad (1.20)$$

satisfying

$$\lim_{t \rightarrow 0} u(x, t) = 0 \quad \text{locally uniformly in } F^c. \quad (1.21)$$

Extension to positive solutions of

$$\partial_t u - \Delta u + f(u) = 0 \quad \text{in } Q_T \quad (1.22)$$

where  $f$  is continuous from  $\mathbb{R}^+$  to  $\mathbb{R}^+$  and satisfies

$$c_2 r^q \leq f(r) \leq c_1 r^q \quad \forall r \geq 0 \quad (1.23)$$

for some  $0 < c_2 \leq c_1$  is straightforward.

This *quasi representation*, up to uniformly upper and lower bounded functions, is also interesting in the sense that it indicates precisely *how to characterize the blow-up points* of  $\overline{u}_F = \underline{u}_F := u_F$ . Introducing an integral expression comparable to  $W_F$ , we show in particular the following results

$$\lim_{\tau \rightarrow 0} \tau^{\frac{2}{q-1}-N} C_{2/q,q'}(F \cap B_\tau(x)) = \gamma \in [0, \infty) \implies \lim_{t \rightarrow 0} t^{\frac{1}{q-1}} u_F(x, t) = C\gamma \quad (1.24)$$

for some  $C_\gamma = C(N, q, \gamma) > 0$ , and

$$\limsup_{\tau \rightarrow 0} \tau^{\frac{2}{q-1}} C_{2/q,q'}\left(\frac{F}{\tau} \cap B_1(x)\right) < \infty \implies \limsup_{t \rightarrow 0} u_F(x, t) < \infty. \quad (1.25)$$

Our paper is organized as follows. In Section 1 we recall some properties of the Besov spaces with fractional derivatives  $B^{s,p}$  and their links with heat equation. In Section 2 we obtain estimates from above on  $\overline{u}_F$ . In Section 3 we give estimates from below on  $\underline{u}_F$ . In Section 4 we prove the main theorems and expose various consequences. In Appendix we derive a series of sharp integral inequalities.

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## 2 Estimates from above

*Some notations.* Let  $\Omega$  be a domain in  $\mathbb{R}^N$  with a compact  $C^2$  boundary and  $T > 0$ . Set  $B_r(a)$  the open ball of radius  $r > 0$  and center  $a$  (and  $B_r(0) := B_r$ ) and

$$Q_T^\Omega := \Omega \times (0, T), \quad \partial_\ell Q_T^\Omega = \partial\Omega \times (0, T), \quad Q_T := Q_T^{\mathbb{R}^N}, \quad Q_\infty := Q_\infty^{\mathbb{R}^N}.$$

Let  $\mathbb{H}^\Omega[\cdot]$  (resp.  $\mathbb{H}[\cdot]$ ) denote the heat potential in  $\Omega$  with zero lateral boundary data (resp. the heat potential in  $\mathbb{R}^N$ ) with corresponding kernel

$$(x, y, t) \mapsto H^\Omega(x, y, t) \quad (\text{resp. } (x, y, t) \mapsto H(x, y, t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}}).$$

We denote by  $q_c := 1 + \frac{N}{2}$ , the Brezis-Friedman critical exponent.

**Theorem 2.1** *Let  $q \geq q_c$ . Then there exists a positive constant  $C_1 = C_1(N, q)$  such that for any closed subset  $F$  of  $\mathbb{R}^N$  and any  $u \in C^2(Q_\infty) \cap C(\overline{Q_\infty} \setminus F)$  satisfying*

$$\begin{aligned} \partial_t u - \Delta u + u^q &= 0 \quad \text{in } Q_\infty \\ \lim_{t \rightarrow 0} u(x, t) &= 0 \quad \text{locally uniformly in } F^c, \end{aligned} \quad (2.1)$$

there holds

$$u(x, t) \leq C_1 W_F(x, t) \quad \forall (x, t) \in Q_\infty, \quad (2.2)$$

where  $W_F$  is the  $(2/q, q')$ -parabolic capacitary potential of  $F$  defined by (1.13).

First we consider the case where  $F = K$  is compact and

$$K \subset B_r \subset \overline{B_r}, \quad (2.3)$$

and then we extend to the general case by a covering argument.

## 2.1 Capacities and Besov spaces

### 2.1.1 $L^p$ regularity

Throughout this paper  $C$  will denote a generic positive constant, depending only on  $N$ ,  $q$  and sometimes  $T$ , the value of which may vary from one occurrence to another. We also use sometimes the notation  $A \approx B$  for meaning that there exists a constant  $C > 0$  independent of the data such that  $C^{-1}A \leq B \leq CA$ .

We recall some classical results dealing with  $L^p$  capacities as they are developed in [5]: if  $1 < p < \infty$  we denote

$$W_p^{2,1}(\mathbb{R}^{N+1}) := \{\phi \in L^p(\mathbb{R}^{N+1}) : \partial_t \phi, \nabla \phi, D^2 \phi \in L^p(\mathbb{R}^{N+1})\}, \quad (2.4)$$

with the associated norm

$$\|\phi\|_{W_p^{2,1}} = \|\phi\|_{L^p} + \|\nabla \phi\|_{L^p} + \|\partial_t \phi\|_{L^p} + \|D^2 \phi\|_{L^p}. \quad (2.5)$$

We define a corresponding capacity on compact sets, that we extend it classically on capacitable sets.

$$C_{2,1,p}(E) = \inf\{\|\phi\|_{W_p^{2,1}} : \phi \in C_0^\infty(\mathbb{R}^{N+1}) : \phi \geq 1 \text{ in a neighborhood of } E\}, \quad (2.6)$$

We extend the heat kernel  $H$  in  $\mathbb{R}^{N+1} = \{(x, t) \in \mathbb{R}^N \times \mathbb{R}\}$  by assigning the value 0 for  $t < 0$ . Then, for any  $\eta \in C_0(\mathbb{R}^N)$ ,

$$\mathbb{H}[\eta](x, t) = \begin{cases} 0 & \text{if } t < 0 \\ H * (\eta \otimes \delta_0)(x, t) & \text{if } t > 0, \end{cases} \quad (2.7)$$

where  $\delta_0$  has to be understood as the Dirac measure on  $\mathbb{R}$  at  $t = 0$ . For any subset  $E \in \mathbb{R}^{N+1}$

$$C_{H,p}(E) = \inf\{\|f\|_{L^p} : f \in L^p(\mathbb{R}^{N+1}), H * f \geq 1 \text{ on } E\}. \quad (2.8)$$

The following result is proved in [5, Prop 2.1].

**Proposition 2.2** *For any  $T > 0$ , there exists  $c = c(T, p, N)$  such that*

$$c^{-1}C_{H,p}(E) \leq C_{2,1,p}(E) \leq cC_{H,p}(E) \quad \forall E \subset \mathbb{R}^N \times ]-T, T[, \text{ } E \text{ Borel}. \quad (2.9)$$

We recall the Gagliardo Nirenberg inequality valid for any  $\phi \in C_0^\infty(\mathbb{R}^d)$

$$\|\nabla \phi\|_{L^{2p}}^{2p} \leq c_{d,p} \|\phi\|_{L^\infty}^p \|D^2 \phi\|_{L^p}^p. \quad (2.10)$$

Furthermore, the trace at  $t = 0$  of functions in  $W_p^{2,1}$  belongs to the Besov space  $B^{2-\frac{2}{p},p}(\mathbb{R}^N)$ . However, in our range of exponents  $B^{2-\frac{2}{p},p}(\mathbb{R}^N) = W^{2-\frac{2}{p},p}(\mathbb{R}^N)$ . The reason for this is that  $2 - \frac{2}{p}$  is not an integer except if  $p = 2$ , in which case equality holds also. If we set

$$c_{2-\frac{2}{p},p}(K) = \inf\{\|\phi\|_{W^{2-\frac{2}{p},p}} : \phi \in C_0^\infty(\mathbb{R}^N), \phi \geq 1 \text{ in a neighborhood of } K\}. \quad (2.11)$$

then [5, Prop 2.3].

**Proposition 2.3** *There exist  $c = c(N, p) > 0$  such that*

$$c^{-1} c_{2-\frac{2}{p},p}(E) \leq C_{2,1,p}(E \times \{0\}) \leq c c_{2-\frac{2}{p},p}(E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel}. \quad (2.12)$$

The  $c_{2-\frac{2}{p},p}$ -capacity is equivalent to the Bessel capacity  $C_{2-\frac{2}{p},p}$  defined by

$$C_{2-\frac{2}{p},p}(E) = \inf\{\|f\|_{L^p} : f \in L^p(\mathbb{R}^N), G_{2-\frac{2}{p}} * f \geq 1 \text{ on } E\} \quad (2.13)$$

where  $G_{2-\frac{2}{p}} = \mathcal{F}[(1+|\xi|^2)^{\frac{1}{p}-1}]$  denotes the Bessel kernel associated to the operator  $(-\Delta + I)^{1-\frac{1}{p}}$ .

### 2.1.2 The Aronszajn-Slobodeckij integral

If  $\Omega$  is a domain in  $\mathbb{R}^N$  and  $0 < s < 1$ , we denote by  $\|\cdot\|_{\dot{B}^{s,p}(\Omega)}$  the Aronszajn-Slobodeckij norm defined on  $C_0^\infty(\Omega)$  by

$$\|\eta\|_{\dot{B}^{s,p}} = \left( \iint_{\Omega \times \Omega} \frac{|\eta(x) - \eta(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{1/p} \quad \forall \eta \in C_0^\infty(\Omega). \quad (2.14)$$

In the case  $1 < s < 2$ , all the results which are presented still holds by replacing the function by its gradient. We also consider the case  $s = 1$ , but in our range of exponents the corresponding exponent for  $p$  is 2, in which case the space under consideration is just  $H_0^1(\Omega)$ . Since the imbedding of  $W^{1,p}(\Omega)$  is compact, it follows the imbedding of  $B^{s,p}(\Omega)$  into  $L^p(\Omega)$  is compact too. Therefore the following Poincaré type inequality holds [39, p. 134]. Actually, the proof, obtained by contradiction, is given with  $W^{1,p}(\Omega)$  instead of  $B^{s,p}(\Omega)$ , but it depends only on the compactness of the imbedding.

**Proposition 2.4** *Let  $\Omega$  be a bounded domain and,  $p \in (1, \infty)$  and  $0 < s \leq 1$  such that  $sp \leq N$ . Then there exists  $\lambda = \lambda(\Omega, N, p) > 0$  such that*

$$\iint_{\Omega \times \Omega} \frac{|\eta(x) - \eta(y)|^p}{|x - y|^{N+sp}} dx dy \geq \lambda \int_{\Omega} |\eta(x)|^p dx \quad \forall \eta \in C_0^\infty(\Omega). \quad (2.15)$$

*Remark.* If  $sp > N$ , the same proof re holds for all  $\eta \in C_0^\infty(\Omega)$  (see the proof of [9, Th 8.2])

$$\left( \iint_{\Omega \times \Omega} \frac{|\eta(x) - \eta(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{1/p} \geq C \frac{|\eta(z) - \eta(z')|}{|z - z'|^\alpha} \quad \forall (z, z') \in \Omega \times \Omega, z \neq z', \quad (2.16)$$



with  $\alpha = s - N/p$  and  $C = C(s, N, p)$ . This estimate implies

$$\left( \iint_{\Omega \times \Omega} \frac{|\eta(x) - \eta(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{1/p} \geq C d^{-\alpha} \|u\|_{L^\infty}, \quad (2.17)$$

where  $d$  is the width of  $\Omega$ , i.e. the smallest of  $\delta > 0$  such that there exists an isometry  $\mathcal{R}$  such that  $\mathcal{R}(\Omega) \subset D_\delta := \{x = (x_1, x') : 0 < x_1 < \delta\}$ .

The related unpublished result due to L. Tartar [40] will be useful in the sequel. We reproduce its proof for the sake of completeness.

**Proposition 2.5** *Assume  $b > a$  and  $\Omega \subset \Gamma_{a,b} := \{x = (x_1, x') : a < x_1 < b\}$  is a domain. If  $sp \leq N$  there exists  $C = C(s, p, N, b/a) > 0$  such that*

$$\iint_{\Omega \times \Omega} \frac{|\eta(x) - \eta(y)|^p}{|x - y|^{N+sp}} dx dy \geq \lambda (b - a)^{sp} \int_\Omega |\eta(x)|^p dx \quad \forall \eta \in C_0^\infty(\Omega). \quad (2.18)$$

*Proof.* Using the notation of [24],  $W^{s,p}(\mathbb{R}^N)$  is the interpolation space  $[W^{1,p}(\mathbb{R}^N), L^p(\mathbb{R}^N)]_{s,p}$  and subset of  $L^p(\mathbb{R}^{N-1}; [W^{1,p}(\mathbb{R}), L^p\mathbb{R}]_{s,p}) = L^p(\mathbb{R}^{N-1}; W^{s,p}(\mathbb{R}))$ , with continuous imbedding. Thus there exist  $C > 0$  such that

$$\begin{aligned} \|\eta\|_{L^p}^p + \int_{\mathbb{R}^{N-1}} \iint_{\mathbb{R} \times \mathbb{R}} \frac{|\eta(x_1, x') - \eta(y_1, x')|^p}{|x_1 - y_1|^{1+sp}} dx_1 dy_1 dx' \\ \leq C \left( \|\eta\|_{L^p}^p + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^p}{|x - y|^{N+sp}} dx dy \right) \end{aligned} \quad (2.19)$$

for all  $\eta \in C_0^\infty(\mathbb{R}^N)$ . This inequality is valid if  $\eta$  is replaced by  $\eta_\tau$  where  $\eta_\tau(x) = \eta(\tau x)$  and  $\tau > 0$ . This gives

$$\begin{aligned} \|\eta\|_{L^p}^p + \tau^{sp-N} \int_{\mathbb{R}^{N-1}} \iint_{\mathbb{R} \times \mathbb{R}} \frac{|\eta(x_1, x') - \eta(y_1, x')|^p}{|x_1 - y_1|^{1+sp}} dx_1 dy_1 dx' \\ \leq C \left( \|\eta\|_{L^p}^p + \tau^{sp-N} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^p}{|x - y|^{N+sp}} dx dy \right). \end{aligned}$$

Letting  $\tau \rightarrow 0$ , we obtained

$$\int_{\mathbb{R}^{N-1}} \iint_{\mathbb{R} \times \mathbb{R}} \frac{|\eta(x_1) - \eta(y_1)|^p}{|x_1 - y_1|^{1+sp}} dx_1 dy_1 dx' \leq C \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^p}{|x - y|^{N+sp}} dx dy \quad \forall \eta \in C_0^\infty(\mathbb{R}^N). \quad (2.20)$$

Using Proposition 2.4 with  $N = 1$  we get

$$\int_0^1 \int_0^1 \frac{|\eta(x_1, x') - \eta(y_1, x')|^p}{|x_1 - y_1|^{1+sp}} dx_1 dy_1 \geq \lambda \int_0^1 |\eta(x_1, x')|^p dx_1 \quad \forall \eta \in C_0^\infty((0, 1) \times \mathbb{R}^{N-1})$$

for all  $x' \in \mathbb{R}^{N-1}$ . Using a standard change of scale, it transforms into

$$\int_a^b \int_a^b \frac{|\eta(x_1, x') - \eta(y_1, x')|^p}{|x_1 - y_1|^{1+sp}} dx_1 dy_1 \geq \lambda (b - a)^{sp} \int_a^b |\eta(x_1, x')|^p dx_1 \quad \forall \eta \in C_0^\infty((a, b) \times \mathbb{R}^{N-1})$$

Integrating over  $\mathbb{R}^{N-1}$  and using (2.20), we derive (2.18).  $\square$

**Definition 2.6** Assume  $s \in (0, 1)$  and  $sp < 1$  or  $s = 1$  and  $p = 2$ . If  $\Omega$  is any domain in  $\mathbb{R}^N$ , the Besov space  $B_0^{s,p}(\Omega)$  is the closure of  $C_0^\infty(\Omega)$  with respect to the norm

$$\|\eta\|_{B^{s,p}} = \|\eta\|_{\dot{B}^{s,p}} + \|\eta\|_{L^p}. \quad (2.21)$$

The following result is derived from Proposition 2.5.

**Corollary 2.7** Let  $b > a > 0$  and  $\Omega$  be an open domain of  $\mathbb{R}^N$  such that  $\Omega \subset B_b \setminus \overline{B_a}$ . Then there exists a constant  $C = C(s, p, N) > 0$  such that for any  $\eta \in C_0^\infty(\Omega)$

$$\|\eta\|_{\dot{B}^{s,p}} \leq \|\eta\|_{B^{s,p}} \leq C(b-a)^{sp} \|\eta\|_{\dot{B}^{s,p}}. \quad (2.22)$$

### 2.1.3 Heat potential and Besov space

If  $\eta \in C_0^\infty(\Omega)$ , we extend it by 0 outside  $\Omega$  and set

$$\|\eta\|_{\tilde{B}^{s,p}} = \left( \iint_{Q_\infty} \left| t^{1-s/2} \partial_t \mathbb{H}[\eta] \right|^p dx \frac{dt}{t} \right)^{1/p} \quad (2.23)$$

It is well known (see e.g. [3]) that the Besov space  $B^{s,p}(\Omega)$  can be defined directly as the space of  $\eta \in L^p(\Omega)$  functions such that  $\|\eta\|_{\dot{B}^{s,p}} < \infty$  or or such that  $\|\eta\|_{\tilde{B}^{s,p}} < \infty$ . It coincides with the the interpolation space  $[W^{2,p}(\Omega), L^p(\Omega)]_{s/2,p}$  (see [24]). Furthermore, there exists  $C = C(s, p, N) > 0$  such that

$$C^{-1} (\|\eta\|_{L^p} + \|\eta\|_{\dot{B}^{s,p}}) \leq \|\eta\|_{L^p} + \|\eta\|_{\tilde{B}^{s,p}} \leq C (\|\eta\|_{L^p} + \|\eta\|_{\dot{B}^{s,p}}) \quad \forall \eta \in B^{s,p}(\Omega). \quad (2.24)$$

**Lemma 2.8** Assume  $0 < s < 1$  and  $1 < p < \infty$  or  $s = 1$  and  $p = 2$ . Then there exists a positive constant  $C$ , depending only on  $s, p, N$ , such that for any domain  $\Omega$ , there holds

$$C^{-1} \|\eta\|_{\dot{B}^{s,p}} \leq \|\eta\|_{\tilde{B}^{s,p}} \leq C \|\eta\|_{\dot{B}^{s,p}} \quad \forall \eta \in C_0^\infty(\Omega). \quad (2.25)$$

*Proof.* Let  $\eta \in C_0^\infty(\mathbb{R}^N)$  and  $\tau > 0$ . Set  $\eta_\tau(x) = \eta(\tau x)$ , then (2.25) applied to  $\eta_\tau$  yields to

$$C^{-1} (\|\eta\|_{L^p} + \tau^s \|\eta\|_{\dot{B}^{s,p}}) \leq (\|\eta\|_{L^p} + \tau^s \|\eta\|_{\tilde{B}^{s,p}}) \leq C (\|\eta\|_{L^p} + \tau^s \|\eta\|_{\dot{B}^{s,p}}).$$

Since it holds for any arbitrary large  $\tau$  and  $\eta \in C_0^\infty(\mathbb{R}^N)$ , (2.25) follows.  $\square$

We denote by  $\mathcal{T}_\Omega(K)$  the set of functions  $\eta \in C_0^\infty(\Omega)$  such that  $0 \leq \eta \leq 1$  and  $\eta = 1$  on  $K$ . If  $\Omega$  is a bounded subset of  $\mathbb{R}^N$ , we define the Besov capacity of a compact set  $K \subset \Omega \subset \mathbb{R}^N$  by

$$R_{s,p}^\Omega(K) = \inf \{ \|\eta\|_{\tilde{B}^{s,p}}^p : \eta \in \mathcal{T}_\Omega(K) \}, \quad (2.26)$$

and the Bessel capacity relative to  $\Omega$  by

$$C_{s,p}^\Omega(K) = \inf \{ \|\eta\|_{B^{s,p}}^p : \eta \in \mathcal{T}_\Omega(K) \}. \quad (2.27)$$

We extend classically this capacity to any capacitable set  $K \subset \Omega$ . This capacity has the following scaling property.

**Lemma 2.9** *For any  $\tau > 0$  and any capacitable set  $K \subset \Omega$ , there holds*

$$R_{s,p}^\Omega(K) = \tau^{N-sp} R_{s,p}^{\tau^{-1}\Omega}(\tau^{-1}K). \quad (2.28)$$

*Furthermore, if  $\Omega \subset B_b \setminus \overline{B_a}$ , there exists  $c = c(b-a, b/a, N, s, p) > 0$  such that*

$$c^{-1}C_{s,p}^\Omega(K) \leq R_{s,p}^\Omega(K) \leq cC_{s,p}^\Omega(K). \quad (2.29)$$

*Finally, if  $K \subset \Omega' \subset \overline{\Omega'} \subset \Omega$ , there exists  $c = c(N, s, p, \text{dist}(\Omega', \Omega^c))$  such that*

$$C_{s,p}(K) \leq C_{s,p}^\Omega(K) \leq cC_{s,p}(K). \quad (2.30)$$

*Proof.* The scaling property (2.28) is clear by change of variable. Estimate (2.29) is a consequence of Definition 2.6 and Proposition 2.5. For the last statement, the left-hand side is obvious. For the right-hand side, consider a smooth nonnegative cut-off function  $\zeta$  which is 1 on  $\overline{\Omega'}$ , has value between 0 and 1 and has compact support in  $\Omega$ . If  $\eta \in \mathcal{T}_{\mathbb{R}^N}(K)$ ,  $\zeta\eta \in \mathcal{T}_\Omega(K)$  and

$$\begin{aligned} \|\zeta\eta\|_{B^{s,p}}^p &= \|\zeta\eta\|_{L^p(\Omega)}^p + \|\zeta\eta\|_{\dot{B}^{s,p}}^p \\ &\leq \|\eta\|_{L^p(\Omega)}^p + \|\eta\|_{\dot{B}^{s,p}}^p + \|\zeta\|_{\dot{B}^{s,\infty}}^p \|\eta\|_{L^p}^p \\ &\leq c \|\eta\|_{B^{s,p}}^p, \end{aligned}$$

where

$$\|\zeta\|_{\dot{B}^{s,\infty}} = \sup_{x \neq y} \frac{|\zeta(x) - \zeta(y)|}{|x - y|^s}$$

and  $c \approx 1 + (\text{dist}(\Omega', \Omega^c))^{-s}$ . The proof follows.  $\square$

In the sequel we assume that  $q \geq q_c$  and we take  $p = q'$  and  $s = 2/q$ . If  $K \subset \Omega$ ,  $\Omega$  is bounded and  $\eta \in \mathcal{T}_\Omega(K)$ , we set

$$R[\eta] = |\nabla \mathbb{H}[\eta]|^2 + |\partial_t \mathbb{H}[\eta]|. \quad (2.31)$$

**Lemma 2.10** *There exists  $C = C(N, q) > 0$  such that for every  $\eta \in \mathcal{T}_\Omega(K)$*

$$\|\eta\|_{\dot{B}^{2/q,q'}}^{q'} \leq \iint_{Q_\infty} (R[\eta])^{q'} dx dt := \|R[\eta]\|_{L^{q'}}^{q'} \leq C \|\eta\|_{\dot{B}^{2/q,q'}}^{q'} \quad (2.32)$$

*Proof.* Using (2.23) and Lemma 2.8, it follows from Corollary 2.7 that

$$\|\eta\|_{\dot{B}^{2/q,q'}}^{q'} \approx \iint_{Q_\infty} |\partial_t \mathbb{H}[\eta]|^{q'} dx dt.$$

Using the Gagliardo-Nirenberg inequality in  $\mathbb{R}^N$ , an elementary elliptic estimate and the fact that  $0 \leq \mathbb{H}[\eta] \leq 1$ , we see that

$$\int_{\mathbb{R}^N} |\nabla(\mathbb{H}[\eta](\cdot, t))|^{2q'} dx \leq C \|D^2 \mathbb{H}[\eta](\cdot, t)\|_{L^{q'}}^{q'} \|\mathbb{H}[\eta](\cdot, t)\|_{L^\infty}^{q'} \leq C \|\Delta \mathbb{H}[\eta](\cdot, t)\|_{L^{q'}}^{q'}, \quad (2.33)$$

for all  $t > 0$ . Since  $\partial_t \mathbb{H}[\eta] = \Delta \mathbb{H}[\eta]$ , it implies (2.32).  $\square$

The dual space  $B^{-2/q,q}(\Omega)$  of  $B^{2/q,q'}(\Omega)$  is naturally endowed with the norm

$$\|\mu\|_{B^{-2/q,q}} = \sup \left\{ \mu(\eta) : \eta \in B^{2/q,q'}(\Omega), \|\eta\|_{B^{2/q,q'}} \leq 1 \right\}.$$

The following result is may be already known, but we have not found it in the literature. If  $\mu$  is a bounded measure in  $\mathbb{R}^N$ , we denote by  $\mathbb{H}[\mu]$  the solution of heat equation in  $Q_\infty$  with initial data  $\mu$ .

**Lemma 2.11** *Assume  $q \geq q_c$ . For any  $T > 0$ , there exist a constant  $c > 0$  such that, for any bounded measure  $\mu$  belonging to  $B^{-2/q,q}(\mathbb{R}^N)$ , there holds*

$$c^{-1} \|\mu\|_{B^{-2/q,q}(\mathbb{R}^N)} \leq \|\mathbb{H}[\mu]\|_{L^q(Q_T)} \leq c \|\mu\|_{B^{-2/q,q}(\mathbb{R}^N)}. \quad (2.34)$$

Furthermore, if  $q > q_c$  there holds

$$c^{-1} \|\mu\|_{B^{-2/q,q}(\mathbb{R}^N)} \leq \|\mathbb{H}[\mu]\|_{L^q(Q_\infty)} \leq c \|\mu\|_{B^{-2/q,q}(\mathbb{R}^N)} + c \|\mu\|_{\mathfrak{M}(\mathbb{R}^N)}. \quad (2.35)$$

*Proof.* If  $\mu \in B^{-2/q,q}(\mathbb{R}^N)$ , there exists a unique  $\omega \in B^{2-2/q,q}(\mathbb{R}^N)$  such that  $\mu = (I - \Delta)\omega$ , and  $\|\mu\|_{B^{-2/q,q}} \approx \|\omega\|_{B^{2-2/q,q}}$ . Applying standard interpolation methods to the analytic semi-group  $e^{-t(I-\Delta)} = e^{-t}e^{t\Delta}$  (see e.g. [3], [41]) we obtain,

$$\begin{aligned} \left( \iint_{Q_\infty} |t^{1/q}(I - \Delta)\mathbb{H}[\omega]|^q dx \frac{e^{-qt}dt}{t} \right)^{1/q} &= \left( \iint_{Q_\infty} |t^{1/q}\mathbb{H}[\mu]|^q dx \frac{e^{-qt}dt}{t} \right)^{1/q} \\ &\approx \|\omega\|_{B^{2-2/q,q}} \\ &\approx \|\mu\|_{B^{-2/q,q}}. \end{aligned} \quad (2.36)$$

Clearly

$$e^{-qT} \iint_{Q_T} |t^{1/q}\mathbb{H}[\mu]|^q dx \frac{dt}{t} \leq \iint_{Q_\infty} |t^{1/q}\mathbb{H}[\mu]|^q dx \frac{e^{-qt}dt}{t},$$

and

$$\begin{aligned} \iint_{Q_\infty} |t^{1/q}\mathbb{H}[\mu]|^q dx \frac{e^{-qt}dt}{t} &= \sum_{n=0}^{\infty} \iint_{Q_{T+n+1} \setminus Q_{T+n}} |t^{1/q}\mathbb{H}[\mu]|^q dx \frac{e^{-qt}dt}{t} \\ &= \sum_{n=0}^{\infty} \iint_{Q_T} |\mathbb{H}[\mu](s+n)|^q e^{-q(s+n)} ds \\ &\leq \left( \sum_{n=0}^{\infty} e^{-qn} \right) \iint_{Q_T} |t^{1/q}\mathbb{H}[\mu]|^q \frac{dt}{t}. \end{aligned}$$

This implies (2.34). Furthermore,  $\|\mathbb{H}[\mu](\cdot, t)\|_{L^q}^q \leq ct^{-N(q-1)/2} \|\mu\|_{\mathfrak{M}}^q$ , thus  $\mathbb{H}[\mu] \in L^q(Q_\infty)$  if  $q > q_c$  (but this does not hold if  $q = q_c$ ). If  $q > q_c$  (equivalently  $N(q-1)/2 > 1$ ),

$$\begin{aligned}
\iint_{Q_\infty} |t^{1/q} \mathbb{H}[\mu]|^q dx \frac{dt}{t} &= \sum_{n=0}^{\infty} \iint_{Q_{T+n+1} \setminus Q_{T+n}} |t^{1/q} \mathbb{H}[\mu]|^q dx \frac{dt}{t} \\
&= \iint_{Q_T} |t^{1/q} \mathbb{H}[\mu]|^q dx \frac{dt}{t} + \iint_{Q_T} \sum_{n=1}^{\infty} |\mathbb{H}[\mu](s+n)|^q dx ds \\
&\leq \iint_{Q_T} |t^{1/q} \mathbb{H}[\mu]|^q dx \frac{dt}{t} + C \left( \sum_{n=1}^{\infty} n^{-N(q-1)/2} \right) \|\mu\|_{\mathfrak{M}}^q.
\end{aligned}$$

Thus we obtain (2.35).  $\square$

## 2.2 Global $L^q$ -estimates

Let  $\rho > 0$ , we assume (2.3) holds. With the previous notations,  $\mathcal{T}_{r,\rho}(K)$  denotes the set of functions  $\eta \in C_0^\infty(B_{r+\rho})$ , such that  $0 \leq \eta \leq 1$  and value 1 on  $K$ . If  $\eta \in \mathcal{T}_{r,\rho}(K)$ , we set

$$\eta^* = 1 - \eta \quad \text{and} \quad \zeta = \mathbb{H}[\eta^*]^{2q'}.$$

**Lemma 2.12** *Assume  $u$  is a positive solution of (2.1) in  $Q_\infty$ . There exists  $C = C(N, q) > 0$  such that for every  $T > 0$  and every compact set  $K \subset B_r$ ,*

$$\iint_{Q_T} u^q \zeta dx dt + \int_{\mathbb{R}^N} (u\zeta)(x, T) dx \leq C \|R[\eta]\|_{L^{q'}}^{q'} \quad \forall \eta \in \mathcal{T}_{r,\rho}(K). \quad (2.37)$$

*Proof.* We recall that there always holds

$$0 \leq u(x, t) \leq \left( \frac{1}{t(q-1)} \right)^{\frac{1}{q-1}} \quad \forall (x, t) \in Q_\infty, \quad (2.38)$$

and

$$0 \leq u(x, t) \leq \left( \frac{C}{t + (|x| - r)^2} \right)^{\frac{1}{q-1}} \quad \forall (x, t) \in Q_\infty \setminus B_r \times \mathbb{R}, \quad (2.39)$$

by the Brezis-Friedman estimate [7]. Since  $\eta^*$  vanishes in an open neighborhood  $\mathcal{N}_1$ , for any open subset  $\mathcal{N}_2$  such that  $K \subset \mathcal{N}_2 \subset \overline{\mathcal{N}_2} \subset \mathcal{N}_1$  there exist  $c_2 = c_{\mathcal{N}_2} > 0$  and  $C_2 = C_{\mathcal{N}_2} > 0$  such that

$$\mathbb{H}[\eta^*](x, t) \leq C_2 e^{-\frac{c_2}{t}}, \quad \forall (x, t) \in Q_T^{\mathcal{N}_2}.$$

Therefore

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} (u\zeta)(x, t) dx = 0.$$

Thus  $\zeta$  is an admissible test function and one has

$$\iint_{Q_T} u^q \zeta dx dt + \int_{\mathbb{R}^N} (u\zeta)(x, T) dx = \iint_{Q_T} u(\partial_t \zeta + \Delta \zeta) dx dt. \quad (2.40)$$

Notice that the two terms on the left-hand side are nonnegative. Put  $\mathbb{H}_{\eta^*} = \mathbb{H}[\eta^*]$ , then

$$\begin{aligned}\partial_t \zeta + \Delta \zeta &= 2q' \mathbb{H}_{\eta^*}^{2q'-1} (\partial_t \mathbb{H}_{\eta^*} + \Delta \mathbb{H}_{\eta^*}) + 2q'(2q' - 1) \mathbb{H}_{\eta^*}^{2q'-2} |\nabla \mathbb{H}_{\eta^*}|^2, \\ &= 2q' \mathbb{H}_{\eta^*}^{2q'-1} (\partial_t \mathbb{H}_{\eta} + \Delta \mathbb{H}_{\eta}) + 2q'(2q' - 1) \mathbb{H}_{\eta}^{2q'-2} |\nabla \mathbb{H}_{\eta}|^2,\end{aligned}$$

because  $\mathbb{H}_{\eta^*} = 1 - \mathbb{H}_{\eta}$ , hence

$$u(\partial_t \zeta + \Delta \zeta) = u \mathbb{H}_{\eta^*}^{2q'/q} \left[ 2q'(2q' - 1) \mathbb{H}_{\eta^*}^{2q'-2-2q'/q} |\nabla \mathbb{H}_{\eta}|^2 - 2q' \mathbb{H}_{\eta^*}^{2q'-1-2q'/q} (\Delta \mathbb{H}_{\eta} + \partial_t \mathbb{H}_{\eta}) \right].$$

Finally, since  $2q' - 2 - 2q'/q = 0$  and  $0 \leq \mathbb{H}_{\eta^*} \leq 1$ , there holds

$$\left| \iint_{Q_T} u(\partial_t \zeta + \Delta \zeta) dx dt \right| \leq C(q) \left( \iint_{Q_T} u^q \zeta dx dt \right)^{1/q} \left( \iint_{Q_T} R^{q'}(\eta) dx dt \right)^{1/q'},$$

where

$$R(\eta) = |\nabla \mathbb{H}_{\eta}|^2 + |\Delta \mathbb{H}_{\eta} + \partial_t \mathbb{H}_{\eta}|.$$

Using Lemma 2.10 one obtains (2.37).  $\square$

**Proposition 2.13** *Under the assumptions of Lemma 2.12, let  $r > 0$ ,  $\rho > 0$ ,  $T \geq (r + \rho)^2$*

$$\mathcal{E}_{r+\rho} := \{(x, t) : |x|^2 + t \leq (r + \rho)^2\}$$

and  $Q_{r+\rho, T} = Q_T \setminus \mathcal{E}_{r+\rho}$ . There exists  $C = C(N, q, T) > 0$  such that

$$\iint_{Q_{r+\rho, T}} u^q dx dt + \int_{\mathbb{R}^N} u(x, T) dx \leq C \|R[\eta]\|_{L^{q'}}^{q'} \quad \forall \eta \in \mathcal{T}_{r, \rho}(K). \quad (2.41)$$

*Proof.* In view of Lemma 2.12 we only have to show that there exists a positive constant  $c(N, q)$  such that, for  $\eta$  as above and  $T \geq (r + \rho)^2$ ,

$$\zeta = \mathbb{H} \eta^{*2q'} > c(N, q).$$

Since, by assumption  $K \subset B_r$ ,  $\eta^* \equiv 1$  outside  $B_{r+\rho}$  and  $0 \leq \eta^* \leq 1$ ,

$$\begin{aligned}\mathbb{H}[\eta^*](x, t) \geq \mathbb{H}[1 - \chi_{B_{r+\rho}}](x, t) &= \left( \frac{1}{4\pi t} \right)^{\frac{N}{2}} \int_{|y| \geq r+\rho} e^{-\frac{|x-y|^2}{4t}} dy, \\ &= 1 - \left( \frac{1}{4\pi t} \right)^{\frac{N}{2}} \int_{|y| \leq r+\rho} e^{-\frac{|x-y|^2}{4t}} dy.\end{aligned}$$

For  $(x, t) \in Q_{r+\rho, T}$ , put  $x = (r + \rho)\xi$ ,  $y = (r + \rho)v$  and  $t = (r + \rho)^2\tau$ . Then  $(\xi, \tau) \in Q_{1, \frac{T}{(r+\rho)^2}}$  and

$$\left( \frac{1}{4\pi t} \right)^{\frac{N}{2}} \int_{|y| \leq r+\rho} e^{-\frac{|x-y|^2}{4t}} dy = \left( \frac{1}{4\pi \tau} \right)^{\frac{N}{2}} \int_{|v| \leq 1} e^{-\frac{|\xi-v|^2}{4\tau}} dv.$$

We claim that

$$\max \left\{ \left( \frac{1}{4\pi\tau} \right)^{\frac{N}{2}} \int_{|v| \leq 1} e^{-\frac{|\xi-v|^2}{4\tau}} dv : (\xi, \tau) \in Q_{1, \frac{T}{(r+\rho)^2}} \right\} = \ell, \quad (2.42)$$

for some  $\ell = \ell(N, \frac{T}{(r+\rho)^2}) \in (0, 1]$ , and  $\ell$  is actually independent of  $\frac{T}{(r+\rho)^2}$  if this quantity is larger than 1. We recall that

$$\left( \frac{1}{4\pi\tau} \right)^{\frac{N}{2}} \int_{|v| \leq 1} e^{-\frac{|\xi-v|^2}{4\tau}} dv < 1 \quad \forall \tau > 0. \quad (2.43)$$

If the maximum is achieved for some  $(\bar{\xi}, \bar{\tau}) \in Q_{1, \frac{T}{(r+\rho)^2}}$ , it is smaller than 1 and

$$\mathbb{H}[\eta^*](x, t) \geq \mathbb{H}[1 - \chi_{B_{r+\rho}}](x, t) \geq 1 - \ell > 0, \quad \forall (x, t) \in Q_{r+\rho, T}. \quad (2.44)$$

Let us assume that the maximum is achieved following a sequence  $\{(\xi_n, \tau_n)\}$  with  $\tau_n \rightarrow 0$  and  $|\xi_n| \rightarrow \alpha \geq 1$ . Then

$$\left( \frac{1}{4\pi\tau_n} \right)^{\frac{N}{2}} \int_{|v| \leq 1} e^{-\frac{|\xi_n-v|^2}{4\tau_n}} dv = \left( \frac{1}{4\pi\tau_n} \right)^{\frac{N}{2}} \int_{B_1(\xi_n)} e^{-\frac{|v|^2}{4\tau_n}} dv \leq \frac{1}{2}.$$

To verify this, note that  $B_1(\xi_n) \cap B_1(-\xi_n) = \emptyset$ , so that

$$\int_{B_1(\xi_n)} e^{-\frac{|v|^2}{4\tau_n}} dv + \int_{B_1(-\xi_n)} e^{-\frac{|v|^2}{4\tau_n}} dv < \int_{\mathbb{R}^N} e^{-\frac{|v|^2}{4\tau_n}} dv < 1$$

and

$$\int_{B_1(\xi_n)} e^{-\frac{|v|^2}{4\tau_n}} dv = \int_{B_1(-\xi_n)} e^{-\frac{|v|^2}{4\tau_n}} dv.$$

If the supremum is achieved with a sequence  $\{(\xi_n, \tau_n)\}$  such that  $|\xi_n| \rightarrow \infty$ , the same argument applies. Finally if  $\{\xi_n\}$  is bounded but  $\tau_n \rightarrow \infty$  then the expression in (2.43) tends to zero. Therefore (2.43) holds. Put  $C = (1 - \ell)^{-1}$ , then

$$\iint_{Q_{r,T}} u^q dx dt + \int_{\mathbb{R}^N} u(\cdot, T) dx \leq C \|R[\eta]\|_{L^{q'}}^{q'}, \quad (2.45)$$

and (2.41) follows.  $\square$

### 2.3 Pointwise estimates

In this subsection  $u$  is a positive solution of (2.1) in  $Q_\infty$  and the assumptions of Lemma 2.12 hold. We first derive a rough pointwise estimate.

**Lemma 2.14** *There exists a constant  $C = C(N, q) > 0$  such that, for any  $\eta \in \mathcal{T}_{r,\rho}(K)$ ,*

$$u(x, (r + 2\rho)^2) \leq \frac{C \|R[\eta]\|_{L^{q'}}^{q'}}{(\rho(r + \rho))^{\frac{N}{2}}}, \quad \forall x \in \mathbb{R}^N. \quad (2.46)$$

*Proof.* We recall that

$$\int_s^T \int_{\mathbb{R}^N} u^q dx dt + \int_{\mathbb{R}^N} u(x, T) dx = \int_{\mathbb{R}^N} u(x, s) dx \quad \forall T > s > 0, \quad (2.47)$$

and

$$\int_{\mathbb{R}^N} u(., s) dx \leq C \|R[\eta]\|_{L^{q'}}^{q'} \quad \forall T > s \geq (r + \rho)^2, \quad (2.48)$$

by Proposition 2.13. Using the fact that

$$u(x, \tau + s) \leq \mathbb{H}[u(., s)](x, \tau) \leq \left( \frac{1}{4\pi\tau} \right)^{\frac{N}{2}} \int_{\mathbb{R}^N} u(., s) dx,$$

(2.46) follows from (2.48) with  $s = (r + \rho)^2$  and  $\tau = (r + 2\rho)^2 - (r + \rho)^2 \approx \rho(r + \rho)$ .  $\square$

The above estimate does not take into account the fact that  $u(x, 0) = 0$  if  $|x| \geq r$ . It is mainly interesting if  $|x| \leq r$ . In order to derive a sharper estimate which takes this fact into account, we need some lateral boundary estimates.

**Lemma 2.15** *Let  $\gamma \geq r + 2\rho$  and  $c > 0$  and either  $N = 1$  or  $2$  and  $0 \leq t \leq c\gamma^2$  for some  $c > 0$ , or  $N \geq 3$  and  $t > 0$ . Then, for any  $\eta \in \mathcal{T}_{r,\rho}(K)$ , there holds*

$$\int_0^t \int_{\partial B_\gamma} u dS d\tau \leq C_5 \gamma \|R[\eta]\|_{L^{q'}}^{q'}. \quad (2.49)$$

where  $C > 0$  depends on  $N$ ,  $q$  and  $c$  if  $N = 1, 2$  or depends only on  $N$  and  $q$  if  $N \geq 3$ .

*Proof.* First we assume  $N = 1$  or  $2$ . Put  $G^\gamma := B_\gamma^c \times (-\infty, 0)$  and  $\partial_\ell G^\gamma = \partial B_\gamma \times (-\infty, 0)$ . We set

$$h_\gamma(x) = 1 - \frac{\gamma}{|x|},$$

and let  $\psi_\gamma$  be the solution of

$$\begin{aligned} \partial_\tau \psi_\gamma + \Delta \psi_\gamma &= 0 & \text{in } G^\gamma, \\ \psi_\gamma &= 0 & \text{on } \partial_\ell G^\gamma, \\ \psi_\gamma(., 0) &= h_\gamma & \text{in } B_\gamma^c. \end{aligned} \quad (2.50)$$

Thus the function

$$\tilde{\psi}(x, \tau) = \psi_\gamma(\gamma x, \gamma^2 \tau)$$

satisfies

$$\begin{aligned} \partial_t \tilde{\psi} + \Delta \tilde{\psi} &= 0 & \text{in } G^1 \\ \tilde{\psi} &= 0 & \text{on } \partial_\ell G^1 \\ \tilde{\psi}(., 0) &= \tilde{h} & \text{in } B_1^c, \end{aligned} \quad (2.51)$$

and  $\tilde{h}(x) = 1 - |x|^{-1}$ . By the maximum principle  $0 \leq \tilde{\psi} \leq 1$ , and by Hopf Lemma

$$-\frac{\partial \tilde{\psi}}{\partial \mathbf{n}} \Big|_{\partial B_1 \times [-c, 0]} \geq \theta > 0, \quad (2.52)$$



where  $\theta = \theta(N, c)$ . Then  $0 \leq \psi_\gamma \leq 1$  and

$$-\frac{\partial \psi_\gamma}{\partial \mathbf{n}} \Big|_{\partial B_\gamma \times [-\gamma^2, 0]} \geq \theta/\gamma. \quad (2.53)$$

Multiplying (1.1) by  $\psi_\gamma(x, \tau - t) = \psi_\gamma^*(x, \tau)$  and integrating on  $B_\gamma^c \times (0, t)$  yields to

$$\int_0^t \int_{B_\gamma^c} u^q \psi_r^* dx d\tau + \int_{B_\gamma^c} (u h_\gamma)(x, t) dx - \int_0^t \int_{\partial B_\gamma} \frac{\partial u}{\partial \mathbf{n}} \psi_\gamma^* dS d\tau = - \int_0^t \int_{\partial B_\gamma} \frac{\partial \psi_\gamma^*}{\partial \mathbf{n}} u d\sigma d\tau. \quad (2.54)$$

Since  $\psi_\gamma^*$  is bounded from above by 1, estimate (2.49) follows from (2.53) and Proposition 2.13 (notice that  $B_\gamma^c \times (0, t) \subset \mathcal{E}_\gamma^c$ ), first by taking  $t = T = \gamma^2 \geq (r + 2\rho)^2$ , and then for any  $t \leq \gamma^2$ .

If  $N \geq 3$ , we proceed as above except that we take

$$h_\gamma(x) = 1 - \left( \frac{\gamma}{|x|} \right)^{N-2}.$$

Then  $\psi_\gamma(x, t) = h_\gamma(x)$  and  $\theta = N - 2$  is independent of the length of the time interval. This leads to the conclusion.  $\square$

**Lemma 2.16** *I- Let  $M, a > 0$  and  $\eta \in L^\infty(\mathbb{R}^N)$  such that*

$$0 \leq \eta(x) \leq M e^{-a|x|^2} \quad \text{a.e. in } \mathbb{R}^N. \quad (2.55)$$

*Then, for any  $t > 0$ ,*

$$0 \leq \mathbb{H}[\eta](x, t) \leq \frac{M}{(4at + 1)^{\frac{N}{2}}} e^{-\frac{a|x|^2}{4at+1}} \quad \forall x \in \mathbb{R}^N. \quad (2.56)$$

*II- Let  $M, a, b > 0$  and  $\eta \in L^\infty(\mathbb{R}^N)$  such that*

$$0 \leq \eta(x) \leq M e^{-a(|x|-b)_+^2} \quad \text{a.e. in } \mathbb{R}^N. \quad (2.57)$$

*Then, for any  $t > 0$ ,*

$$0 \leq \mathbb{H}[\eta](x, t) \leq \frac{M e^{-\frac{a(|x|-b)_+^2}{4at+1}}}{(4at + 1)^{\frac{N}{2}}} \quad \forall x \in \mathbb{R}^N, \forall t > 0. \quad (2.58)$$

*Proof.* For the first statement, put  $a = \frac{1}{4}s$ . Then

$$0 \leq \eta(x) \leq M(4\pi s)^{\frac{N}{2}} \frac{1}{(4\pi s)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4s}} = C(4\pi s)^{\frac{N}{2}} \mathbb{H}[\delta_0](x, s).$$

By the order property of the heat kernel,

$$0 \leq \mathbb{H}[\eta](x, t) \leq M(4\pi s)^{\frac{N}{2}} \mathbb{H}[\delta_0](x, t + s) = M \left( \frac{s}{t + s} \right)^{\frac{N}{2}} e^{-\frac{|x|^2}{4(t+s)}},$$

and (2.56) follows by replacing  $s$  by  $\frac{1}{4}a$ .

For the second statement, let  $\tilde{a} < a$  and  $R = \max\{e^{-a(r-b)_+^2 + \tilde{a}r^2} : r \geq 0\}$ . A direct computation gives  $R = e^{\frac{a\tilde{a}b^2}{a-\tilde{a}}}$ , and (2.58) implies

$$0 \leq \eta(x) \leq Me^{\frac{a\tilde{a}b^2}{a-\tilde{a}}} e^{-\tilde{a}|x|^2}.$$

Applying the statement I, we derive

$$0 \leq \mathbb{H}[\eta](x, t) \leq \frac{Ce^{\frac{a\tilde{a}b^2}{a-\tilde{a}}}}{(4\tilde{a}t + 1)^{\frac{N}{2}}} e^{-\frac{\tilde{a}|x|^2}{4\tilde{a}t+1}} \quad \forall x \in \mathbb{R}^N, \quad \forall t > 0. \quad (2.59)$$

Since for any  $x \in \mathbb{R}^N$  and  $t > 0$ ,

$$(4\tilde{a}t + 1)^{-\frac{N}{2}} e^{-\frac{\tilde{a}|x|^2}{4\tilde{a}t+1}} \leq e^{-\frac{a\tilde{a}b^2}{a-\tilde{a}}} (4at + 1)^{-\frac{N}{2}} e^{-\frac{a(|x|-b)^2}{4at+1}},$$

(2.58) follows from (2.59).  $\square$

**Lemma 2.17** *There exists a constant  $C = C(N, q) > 0$  such that, for any  $\eta \in \mathcal{T}_{r,\rho}(K)$ , there holds*

$$u(x, (r + 2\rho)^2) \leq C \max \left\{ \frac{r + \rho}{(|x| - r - 2\rho)^{N+1}}, \frac{|x| - r - 2\rho}{(r + \rho)^{N+1}} \right\} e^{-\frac{(|x| - (r+2\rho))^2}{4(r+2\rho)^2}} \|R[\eta]\|_{L^{q'}}, \quad (2.60)$$

for any  $x \in \mathbb{R}^N \setminus B_{r+3\rho}$ .

*Proof.* It is classical that the Dirichlet heat kernel  $H^{B_1^c}$  in the complement of  $B_1$  satisfies, for some  $C = C(N) > 0$ ,

$$H^{B_1^c}(x', y', t', s') \leq C_7 (t' - s')^{-(N+2)/2} (|x'| - 1) e^{-\frac{|x' - y'|^2}{4(t' - s')}}, \quad (2.61)$$

for  $t' > s'$ . By performing the change of variable  $x' \mapsto (r + 2\rho)x'$ ,  $t' \mapsto (r + 2\rho)^2 t'$ , for any  $x \in \mathbb{R}^N \setminus B_{r+2\rho}$  and  $0 \leq t \leq T$ , one obtains

$$u(x, t) \leq C(|x| - r - 2\rho) \int_0^t \int_{\partial B_{r+2\rho}} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{1+\frac{N}{2}}} u(y, s) d\sigma(y) ds. \quad (2.62)$$

The right-hand side term in (2.62) is smaller than

$$\max \left\{ \frac{C(|x| - r - 2\rho)}{(t-s)^{1+\frac{N}{2}}} e^{-\frac{(|x|-r-2\rho)^2}{4(t-s)}} : s \in (0, t) \right\} \int_0^t \int_{\partial B_{r+2\rho}} u(y, s) d\sigma(y) ds.$$

We fix  $t = (r + 2\rho)^2$  and  $|x| \geq r + 3\rho$ . Since

$$\begin{aligned} & \max \left\{ \frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : s \in (0, (r+2\rho)^2) \right\} \\ &= (|x| - r - 2\rho)^{-2-N} \max \left\{ \frac{e^{-\frac{1}{4\sigma}}}{\sigma^{1+\frac{N}{2}}} : 0 < \sigma < \left( \frac{r+2\rho}{|x| - r - 2\rho} \right)^2 \right\}, \end{aligned}$$

a direct computation gives

$$\begin{aligned} & \max \left\{ \frac{e^{-\frac{1}{4}\sigma}}{\sigma^{1+\frac{N}{2}}} : 0 < \sigma < \left( \frac{r+2\rho}{|x|-r-2\rho} \right)^2 \right\} \\ &= \begin{cases} (2N+4)^{1+\frac{N}{2}} e^{-(N+2)/2} & \text{if } r+3\rho \leq |x| \leq (r+2\rho)(1+\sqrt{4+2N}), \\ \left( \frac{|x|-r-2\rho}{r+2\rho} \right)^{2+N} e^{-\left( \frac{|x|-r-2\rho}{2r+4\rho} \right)^2} & \text{if } |x| \geq (r+2\rho)(1+\sqrt{4+2N}). \end{cases} \end{aligned}$$

Thus there exists a constant  $C(N) > 0$  such that

$$\max \left\{ \frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : s \in (0, (r+2\rho)^2) \right\} \leq C(N) \rho^{-2-N} e^{-\left( \frac{|x|-(r+2\rho)}{2r+4\rho} \right)^2}. \quad (2.63)$$

Combining this estimate with (2.49) with  $\gamma = r+2\rho$  and (2.62), one derives (2.60).  $\square$

**Lemma 2.18** *There exists a constant  $C = C(N, q) > 0$  such that*

$$0 \leq u(x, (r+2\rho)^2) \leq C \max \left\{ \frac{(r+\rho)^3}{\rho(|x|-r-2\rho)^{N+1}}, \frac{1}{(r+\rho)^{N-1}\rho} \right\} e^{-\left( \frac{|x|-r-3\rho}{2r+4\rho} \right)^2} \|R[\eta]\|_{L^{q'}}^{q'}, \quad (2.64)$$

for every  $x \in \mathbb{R}^N \setminus B_{r+3\rho}$ .

*Proof.* This is a direct consequence of the inequality

$$(|x|-r-2\rho)e^{-\left( \frac{|x|-r-2\rho}{2r+4\rho} \right)^2} \leq \frac{C(r+\rho)^2}{\rho} e^{-\left( \frac{|x|-r-3\rho}{2r+4\rho} \right)^2}, \quad \forall x \in B_{r+2\rho}^c, \quad (2.65)$$

and Lemma 2.17.  $\square$

**Lemma 2.19** *There exists a constant  $C = C(N, q) > 0$  such that, for any  $\eta \in \mathcal{T}_{r,\rho}(K)$ , the following estimate holds*

$$u(x, t) \leq \frac{C\tilde{M}e^{-\frac{(|x|-r-3\rho)_+^2}{4t}}}{t^{\frac{N}{2}}} \|R[\eta]\|_{L^{q'}}^{q'}, \quad \forall x \in \mathbb{R}^N, \forall t \geq (r+2\rho)^2, \quad (2.66)$$

where

$$\tilde{M} = \tilde{M}(x, r, \rho) = \begin{cases} \left(1 + \frac{r}{\rho}\right)^{\frac{N}{2}} & \text{if } |x| < r+3\rho \\ \frac{(r+\rho)^{N+3}}{\rho(|x|-r-2\rho)^{N+2}} & \text{if } r+3\rho \leq |x| \leq c_N^*(r+2\rho) \\ 1 + \frac{r}{\rho} & \text{if } |x| \geq c_N^*(r+2\rho) \end{cases} \quad (2.67)$$

with  $c_N^* = 1 + \sqrt{4+2N}$ .

*Proof.* It follows by the maximum principle

$$u(x, t) \leq \mathbb{H}[u(\cdot, (r + 2\rho)^2)](x, t - (r + 2\rho)^2).$$

for  $t \geq (r + 2\rho)^2$  and  $x \in \mathbb{R}^N$ . By Lemma 2.14 and Lemma 2.18

$$u(x, (r + 2\rho)^2) \leq C_{10} \tilde{M} e^{-\frac{(|x| - r - 3\rho)^2}{4(r + 2\rho)^2}} \|R[\eta]\|_{L^{q'}}^{q'},$$

where

$$\tilde{M} = \begin{cases} ((r + \rho)\rho)^{-\frac{N}{2}} & \text{if } |x| < r + 3\rho \\ \frac{(r + \rho)^3}{\rho} (|x| - r - 2\rho)^{N+2} & \text{if } r + 3\rho \leq |x| \leq c_N^*(r + 2\rho) \\ \frac{1}{(r + \rho)^{N-1}\rho} & \text{if } |x| \geq c_N^*(r + 2\rho) \end{cases}$$

Applying Lemma 2.16 with  $a = (2r + 4\rho)^{-2}$ ,  $b = r + 3\rho$  and  $t$  replaced by  $t - (r + 2\rho)^2$  implies

$$u(x, t) \leq C \frac{(r + 2\rho)^N \tilde{M}}{t^{\frac{N}{2}}} e^{-\frac{(|x| - r - 3\rho)^2}{4t}} \|R[\eta]\|_{L^{q'}}^{q'}, \quad (2.68)$$

for all  $x \in B_{r+3\rho}^c$  and  $t \geq (r + 2\rho)^2$ , which is (2.66).  $\square$

The next estimate gives a precise upper bound for  $u$  when  $t$  is not bounded from below.

**Lemma 2.20** *Assume that  $0 < t \leq (r + 2\rho)^2$ , then there exists a constant  $C = C(N, q) > 0$  such that the following estimate holds*

$$u(x, t) \leq C(r + \rho) \max \left\{ \frac{1}{(|x| - r - 2\rho)^{N+1}}, \frac{1}{\rho t^{\frac{N}{2}}} \right\} e^{-\frac{(|x| - r - 3\rho)^2}{4t}} \|R[\eta]\|_{L^{q'}}^{q'}, \quad (2.69)$$

for any  $(x, t) \in \mathbb{R}^N \setminus B_{r+3\rho} \times (0, (r + 2\rho)^2]$ .

*Proof.* Thanks to (2.49) the following estimate is a straightforward variant of (2.60) for any  $|x| \geq r + 2\rho$ ,

$$u(x, t) \leq C_8(|x| - r - 2\rho)(r + 2\rho) \max \left\{ \frac{e^{-\frac{(|x| - r - 2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : 0 < s \leq t \right\} \|R[\eta]\|_{L^{q'}}^{q'}. \quad (2.70)$$

Clearly

$$\begin{aligned} & \max \left\{ \frac{e^{-\frac{(|x| - r - 2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : 0 < s \leq t \right\} \\ &= \begin{cases} (2N + 4)^{1+\frac{N}{2}} (|x| - r - 2\rho)^{-N-2} e^{-\frac{N+2}{2}} & \text{if } 0 < |x| \leq r + 2\rho + \sqrt{2t(N+2)} \\ \frac{e^{-\frac{(|x| - r - 2\rho)^2}{4t}}}{t^{1+\frac{N}{2}}} & \text{if } |x| > r + 2\rho + \sqrt{2t(N+2)}. \end{cases} \end{aligned}$$

By elementary analysis, if  $x \in B_{r+3\rho}^c$ ,

$$(|x| - r - 2\rho)e^{-\frac{(|x|-r-2\rho)^2}{4t}} \leq e^{-\frac{(|x|-r-3\rho)^2}{4t}} \begin{cases} \rho e^{-\frac{\rho^2}{4t}} & \text{if } 2t < \rho^2 \\ \frac{2t}{\rho} e^{-1+\frac{\rho^2}{4t}} & \text{if } \rho^2 \leq 2t \leq 2(r+2\rho)^2. \end{cases}$$

However, since

$$\frac{\rho}{t} e^{-\frac{\rho^2}{4t}} \leq \frac{4}{\rho},$$

we derive

$$(|x| - r - 2\rho)e^{-\frac{(|x|-r-2\rho)^2}{4t}} \leq \frac{Ct}{\rho} e^{-\frac{(|x|-r-3\rho)^2}{4t}},$$

and (2.69) follows.  $\square$

*Remark.* In the subcritical case  $1 < q < q_c$ , it is easy to show by using Lemma 2.20, that any positive solution  $u$  of (2.1), such that  $u(x, 0) = 0$  for  $x \neq 0$ , satisfies

$$u(x, t) \leq Ct^{-\frac{1}{q-1}} \min \left\{ 1, \left( \frac{|x|}{\sqrt{t}} \right)^{\frac{2}{q-1}-N} e^{-\frac{|x|^2}{4t}} \right\} \quad \forall (x, t) \in Q_\infty. \quad (2.71)$$

This upper estimate corresponds to the one obtained in [8]. If  $F = \overline{B}_r$  the upper estimate is less esthetic. However, it is proved in [28] by a barrier method that, if the initial trace of positive solution  $u$  of (2.1), vanishes outside  $F$ , and if  $1 < q < 3$ , there holds

$$u(x, t) \leq t^{-\frac{1}{q-1}} f_1((|x| - r)/\sqrt{t}) \quad \forall (x, t) \in Q_\infty, |x| \geq r, \quad (2.72)$$

where  $f = f_1$  is the unique positive (and radial) solution of

$$\begin{cases} f'' + \frac{y}{2}f' + \frac{1}{q-1}f - f^q = 0 & \text{in } (0, \infty) \\ f'(0) = 0, \lim_{y \rightarrow \infty} |y|^{\frac{2}{q-1}} f(y) = 0. \end{cases} \quad (2.73)$$

Notice that the existence of  $f_1$  follows from [8] since  $q$  belongs to the subcritical range on exponents in dimension one. Furthermore  $f_1$  has the following asymptotic expansion

$$f_1(y) = Cy^{(3-q)/(q-1)} e^{-y^2/4t} (1 + o(1)) \quad \text{as } y \rightarrow \infty.$$

## 2.4 The upper Wiener test

**Definition 2.21** We define on  $\mathbb{R}^N \times \mathbb{R}$  the two *parabolic distances*  $\delta_2$  and  $\delta_\infty$  by

$$\delta_2[(x, t), (y, s)] := \sqrt{|x - y|^2 + |t - s|}, \quad (2.74)$$

and

$$\delta_\infty[(x, t), (y, s)] := \max\{|x - y|, \sqrt{|t - s|}\}. \quad (2.75)$$

If  $K \subset \mathbb{R}^N$  and  $i = 2, \infty$ ,

$$\delta_i[(x, t), K] = \inf\{\delta_i[(x, t), (y, 0)] : y \in K\} = \begin{cases} \max\{\text{dist}(x, K), \sqrt{|t|}\} & \text{if } i = \infty, \\ \sqrt{\text{dist}^2(x, K) + |t|} & \text{if } i = 2. \end{cases}$$

For  $\beta > 0$  and  $i = 2, \infty$ , we denote by  $\mathcal{B}_\beta^i(m)$  the parabolic ball of center  $m = (x, t)$  and radius  $\beta$  in the parabolic distance  $\delta_i$ .

Let  $K$  be *any* compact subset of  $\mathbb{R}^N$  and  $\bar{u}_K$  the maximal solution of (1.1) which blows up on  $K$ . The function  $\bar{u}_K$  is constructed in [28] as being the decreasing limit of the  $\bar{u}_{K_\epsilon}$  ( $\epsilon > 0$ ) when  $\epsilon \rightarrow 0$ , where

$$K_\epsilon = \{x \in \mathbb{R}^N : \text{dist}(x, K) \leq \epsilon\}$$

and  $\bar{u}_{K_\epsilon} = \lim_{k \rightarrow \infty} u_{k, K_\epsilon} = \bar{u}_K$ , where  $u_k$  is the solution of the classical problem,

$$\begin{cases} \partial_t u_k - \Delta u_k + u_k^q = 0 & \text{in } Q_T, \\ u_k = 0 & \text{on } \partial_t Q_T, \\ u_k(\cdot, 0) = k\chi_{K_\epsilon} & \text{in } \mathbb{R}^N. \end{cases} \quad (2.76)$$

If  $(x, t) = m \in \mathbb{R}^N \times (0, T]$ , we set  $d_K = \text{dist}(x, K)$ ,  $D_K = \max\{|x - y| : y \in K\}$  and  $\lambda = \sqrt{d_K^2 + t} = \delta_2[m, K]$ . We define a slicing of  $K$ , by setting  $d_n = d_n(K, t) := \sqrt{nt}$  ( $n \in \mathbb{N}$ ),  $d_n^\pm = \left(\sqrt{nt} \pm \frac{\sqrt{t}}{\sqrt{n}}\right)_+$  (the positive part is only needed when  $n = 0$ ) and

$$T_n^* = \overline{B}_{d_{n+1}^+}(x) \setminus B_{d_n^-}(x), \quad T_n = \overline{B}_{d_{n+1}}(x) \setminus B_{d_n}(x), \quad \forall n \in \mathbb{N},$$

thus  $T_0^* = B_{2\sqrt{t}}(x)$ ,  $T_0 = B_{\sqrt{t}}(x)$ , and

$$K_n(x, t) = K \cap T_n(x, t) \quad \text{for } n \in \mathbb{N} \text{ and } \mathcal{Q}_n(x, t) = K \cap B_{d_{n+1}}(x, t).$$

When there is no ambiguity, we will skip the  $(x, t)$  variable in the above sets. The main result of this section is the following discrete upper Wiener-type estimate.

**Theorem 2.22** *Assume  $q \geq q_c$ . Then there exists  $C = C(N, q, T) > 0$  such that*

$$\bar{u}_K(x, t) \leq \frac{C}{t^{\frac{N}{2}}} \sum_{n=0}^{a_t} d_{n+1}^{N - \frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q, q'} \left( \frac{K_n}{d_{n+1}} \right) \quad \forall (x, t) \in Q_T, \quad (2.77)$$

where  $a_t$  is the largest integer  $j$  such that  $K_j \neq \emptyset$ .

With no loss of generality, we can assume that  $x = 0$ . Furthermore, in considering the scaling transformation  $u_\ell(y, t) = \ell^{\frac{1}{q-1}} u(\sqrt{\ell}y, \ell t)$ , with  $\ell > 0$ , we can assume  $t = 1$ . Thus the new compact singular set of the initial trace becomes  $K/\sqrt{\ell}$ , that we still denote  $K$ . We also set  $a_K = a_{K,1}$ . For  $n \in \mathbb{N}_*$  set  $\delta_n = d_{n+1} - d_n$ , then  $\frac{1}{2\sqrt{n+1}} \leq \delta_n \leq \frac{1}{2\sqrt{n}}$ . By convention  $\delta_0 = 1$ . It

is possible to exhibit a collection  $\Theta_n$  of points  $a_{n,j}$  with center on the sphere  $\Sigma_n = \{y \in \mathbb{R}^N : |y| = (d_{n+1} + d_n)/2\}$ , such that

$$T_n \subset \bigcup_{a_{n,j} \in \Theta_n} B_{\delta_n}(a_{n,j}), \quad |a_{n,j} - a_{n,k}| \geq \delta_n \quad \text{and} \quad \#\Theta_n \leq Cn^{N-1},$$

for some constant  $C = C(N)$ . If  $K_{n,j} = K_n \cap B_{\delta_n}(a_{n,j})$ , there holds

$$K = \bigcup_{0 \leq n \leq a_K} \bigcup_{a_{n,j} \in \Theta_n} K_{n,j}.$$

The first intermediate step is based on the *quasi-additivity* property of capacities developed in [2].

**Lemma 2.23** *Let  $q \geq q_c$ . There exists a constant  $C = C(N, q)$  such that*

$$\sum_{a_{n,j} \in \Theta_n} R_{2/q, q'}^{B_{2\delta_n}(a_{n,j})}(K_{n,j}) \leq C d_{n+1}^{N - \frac{2}{q-1}} C_{2/q, q'} \left( \frac{K_n}{d_{n+1}} \right) \quad \forall n \in \mathbb{N}_*. \quad (2.78)$$

*Proof.* The following result is proved in [2, Th 3]: if the spheres  $B_{\rho_j^\theta}(b_j)$ ,  $\theta = 1 - 2/N(q-1)$ , are disjoint in  $\mathbb{R}^N$  and  $G$  is an analytic subset of  $\bigcup B_{\rho_j}(b_j)$  where the  $\rho_j$  are positive and smaller than some  $\rho^* > 0$ , there holds

$$C_{2/q, q'}(G) \leq \sum_j C_{2/q, q'}(G \cap B_{\rho_j}(b_j)) \leq A C_{2/q, q'}(G), \quad (2.79)$$

for some  $A$  depending on  $N$ ,  $q$  and  $\rho^*$ . This property is called *quasi-additivity*. We define for  $n \in \mathbb{N}_*$ ,

$$\tilde{T}_n = d_{n+1} T_n, \quad \tilde{K}_n = d_{n+1} K_n \quad \text{and} \quad \tilde{\mathcal{Q}}_n = d_{n+1} \mathcal{Q}_n.$$

Since  $K_{n,j} \subset B_{\delta_n}(a_{n,j})$ , it follows that

$$\tilde{K}_{n,j} := d_{n+1} K_{n,j} \subset B_{d_{n+1}\delta_n}(\tilde{a}_{n,j}).$$

Note that by Lemma 2.9

$$\begin{aligned} R_{2/q, q'}^{B_{2\delta_n}(a_{n,j})}(K_{n,j}) &= d_{n+1}^{\frac{2}{q-1} - N} R_{2/q, q'}^{B_{2\delta_n d_{n+1}}(d_{n+1} a_{n,j})}(\tilde{K}_{n,j}) \\ &\approx d_{n+1}^{\frac{2}{q-1} - N} C_{2/q, q'}^{B_{2\delta_n d_{n+1}}(d_{n+1} a_{n,j})}(\tilde{K}_{n,j}) \\ &\approx d_{n+1}^{\frac{2}{q-1} - N} C_{2/q, q'}(\tilde{K}_{n,j}) \end{aligned} \quad (2.80)$$

where  $\tilde{K}_{n,j} = d_{n+1} K_{n,j}$ . For a fixed  $n > 0$  and each repartition  $\Lambda$  of points  $\tilde{a}_{n,j} = d_{n+1} a_{n,j}$  such that the balls  $B_{2\theta}(\tilde{a}_{n,j})$  are disjoint, the quasi-additivity property holds: if we set

$$K_{n,\Lambda} = \bigcup_{a_{n,j} \in \Lambda} K_{n,j}, \quad \tilde{K}_{n,\Lambda} = d_{n+1} K_{n,\Lambda} = \bigcup_{a_{n,j} \in \Lambda} \tilde{K}_{n,j} \quad \text{and} \quad \tilde{K}_n = d_{n+1} K_n,$$

then

$$\sum_{a_{n,j} \in \Lambda} C_{2/q,q'}(\tilde{K}_{n,j}) \approx C_{2/q,q'}(\tilde{K}_{n,\Lambda}). \quad (2.81)$$

The maximal cardinal of any such repartition  $\Lambda$  is of the order of  $Cn^{N-1}$  for some positive constant  $C = C(N)$ , therefore, the number of repartitions needed for a full covering of the set  $\tilde{T}_n$  is of finite order depending upon the dimension. Because  $\tilde{K}_n$  is the union of the  $\tilde{K}_{n,\Lambda}$ ,

$$\sum_{a_{n,j} \in \Theta_n} C_{2/q,q'}(\tilde{K}_{n,j}) = \sum_{\Lambda} \sum_{a_{n,j} \in \Lambda} C_{2/q,q'}(\tilde{K}_{n,j}) \approx C_{2/q,q'}(\tilde{K}_n). \quad (2.82)$$

By Lemma 2.9,

$$C_{2/q,q'}(\tilde{K}_n) \leq C_{2/q,q'}^{B_{2d_{n+1}}}(\tilde{K}_n) \approx d_{n+1}^{N-\frac{1}{q-1}} C_{2/q,q'}^{B_2} \left( \frac{K_n}{d_{n+1}} \right) \approx d_{n+1}^{N-\frac{1}{q-1}} C_{2/q,q'} \left( \frac{K_n}{d_{n+1}} \right),$$

we obtain (2.78) by combining this last inequality with (2.80) and (2.82).  $\square$

*Proof of Theorem 2.22. Step 1.* We first notice that

$$\bar{u}_K \leq \sum_{0 \leq n \leq a_K} \sum_{a_{n,j} \in \Theta_n} \bar{u}_{K_{n,j}}. \quad (2.83)$$

Actually, since  $K = \bigcup_n \bigcup_{a_{n,j}} K_{n,j}$ , for any  $0 < \epsilon' < \epsilon$ , there holds  $\overline{K_{\epsilon'}} \subset \bigcup_n \bigcup_{a_{n,j}} K_{n,j} \epsilon$ . Because a finite sum of positive solutions of (1.1) is a super solution,

$$\bar{u}_{K_{\epsilon'}} \leq \sum_{0 \leq n \leq a_K} \sum_{a_{n,j} \in \Theta_n} \bar{u}_{K_{n,j} \epsilon}. \quad (2.84)$$

Letting successively  $\epsilon'$  and  $\epsilon$  go to 0 implies (2.83).

*Step 2.* Let  $n \in \mathbb{N}$ . Since  $K_{n,j} \subset B_{\delta_n}(a_{n,j})$  and  $|x - a_{n,j}| = (d_n + d_{n+1})/2$ , we can apply the previous lemmas with  $r = \delta_n$  and  $\rho = r$ . For  $n \geq n_N$ , there holds  $t = 1 \geq (r + 2\rho)^2 = 9/(n+1)$  and  $|x - a_{n,j}| = (\sqrt{n+1} - \sqrt{n})/2 \geq (2 + C_N)(3/\sqrt{n+1})$  (notice that  $n_N \geq 8$ ). Thus

$$u_{K_{n,j}}(0, 1) \leq C e^{(\sqrt{n}-3/\sqrt{n+1})^2/4} R_{2/q,q'}^{B_{2\delta_n}(a_{n,j})}(K_{n,j}) \leq C e^{3/2} e^{-\frac{n}{4}} R_{2/q,q'}^{B_{2\delta_n}(a_{n,j})}(K_{n,j}). \quad (2.85)$$

Using Lemma 2.23 we obtain, with  $d_n = d_n(1) = \sqrt{n+1}$

$$\sum_{n=n_N}^{a_K} \sum_{a_{n,j} \in \Theta_n} u_{K_{n,j}}(0, 1) \leq C \sum_{n=n_N}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'} \left( \frac{K_n}{d_{n+1}} \right). \quad (2.86)$$

Finally, we apply Lemma 2.14 if  $1 \leq n < n_N$  and get

$$\begin{aligned} \sum_1^{n_N-1} \sum_{a_{n,j} \in \Theta_n} u_{K_{n,j}}(0, 1) &\leq C \sum_1^{n_N-1} C_{2/q,q'} \left( \frac{K_n}{d_{n+1}} \right) \\ &\leq C' \sum_1^{n_N-1} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'} \left( \frac{K_n}{d_{n+1}} \right). \end{aligned} \quad (2.87)$$



For  $n = 0$ , we proceed similarly, in splitting  $K_1$  in a finite number of  $K_{1,i}$ , depending only on the dimension, such that  $\text{diam } K_{1,i} < 1/3$ . Combining (2.86) and (2.87), we derive

$$\bar{u}_K(0, 1) \leq C \sum_{n=0}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q, q'} \left( \frac{K_n}{d_{n+1}} \right). \quad (2.88)$$

In order to derive the same result for any  $t > 0$ , we notice that

$$\bar{u}_K(y, t) = t^{-\frac{1}{q-1}} \bar{u}_{K/\sqrt{t}}(y/\sqrt{t}, 1).$$

Going back to the definition of  $d_n = d_n(K, t) = \sqrt{nt} = d_n(K\sqrt{t}, 1)$ , we derive from (2.88) and the fact that  $a_{K,t} = a_{K\sqrt{t}, 1}$

$$\bar{u}_K(0, t) \leq Ct^{-\frac{N}{2}} \sum_{n=0}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q, q'} \left( \frac{K_n}{d_{n+1}} \right), \quad (2.89)$$

with  $d_n = d_n(t) = \sqrt{t(n+1)}$ . This is (2.77) with  $x = 0$ , and a space translation leads to the final result.  $\square$

*Proof of Theorem 2.1.* Let  $m > 0$  and  $F_m = F \cap \bar{B}_m$ . We denote by  $U_{B_m^c}$  the maximal solution of (1.1) in  $Q_\infty$  the initial trace of which vanishes on  $B_m$ . Such a solution is actually the unique solution of (2.1) which satisfies

$$\lim_{t \rightarrow 0} u(x, t) = \infty$$

uniformly on  $B_{m'}^c$ , for any  $m' > m$ : this can be easily proved by noticing that

$$U_{B_m^c}(y, t) = \ell^{\frac{1}{q-1}} U_{B_m^c}(\sqrt{\ell}y, \ell t) = U_{B_{m/\sqrt{\ell}}^c}(y, t).$$

Furthermore

$$\lim_{m \rightarrow \infty} U_{B_m^c}(y, t) = \lim_{m \rightarrow \infty} m^{-\frac{2}{q-1}} U_{B_1^c}(y/m, t/m^2) = 0$$

uniformly on any compact subset of  $\bar{Q}_\infty$ . Since  $\bar{u}_{F_m} + U_{B_m^c}$  is a super-solution, it is larger than  $\bar{u}_F$  and therefore  $\bar{u}_{F_m} \uparrow \bar{u}_F$ . Because  $W_{F_m}(x, t) \leq W_F(x, t)$  and  $\bar{u}_{F_m} \leq C_1 W_{F_m}(x, t)$ , the result follows.  $\square$

*Remark.* It is clear that Theorem 2.1 still holds if  $u$  is a positive subsolution of (1.1) satisfying the initial trace condition (1.21).

Theorem 2.1 admits the following integral expression.

**Theorem 2.24** *Assume  $q \geq q_c$ . Then there exists a positive constant  $C_1^* = C^*(N, q, T)$  such that, for any closed subset  $F$  of  $\mathbb{R}^N$ , there holds*

$$\bar{u}_F(x, t) \leq \frac{C_1^*}{t^{1+\frac{N}{2}}} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} e^{-\frac{s^2}{4t}} s^{N-\frac{2}{q-1}} C_{2/q, q'} \left( \frac{1}{s} F \cap B_1(x) \right) s \, ds, \quad (2.90)$$

where  $a_t = \min\{n : F \subset B_{\sqrt{(n+1)t}}(x)\}$ .

*Proof.* We first use

$$C_{2/q,q'} \left( \frac{F_n}{d_{n+1}} \right) \leq C_{2/q,q'} \left( \frac{F}{d_{n+1}} \cap B_1 \right),$$

and we denote

$$\Phi(s) = C_{2/q,q'} \left( \frac{F}{s} \cap B_1 \right) \quad \forall s > 0. \quad (2.91)$$

*Step 1.* The following inequality holds

$$c_1 \Phi(\alpha s) \leq \Phi(s) \leq c_2 \Phi(\beta s) \quad \forall s > 0, \quad \forall 1/2 \leq \alpha \leq 1 \leq \beta \leq 2, \quad (2.92)$$

for some positive constants  $c_1, c_2$  depending on  $N$  and  $q$ . See [1] and [32]. If  $\beta \in [1, 2]$ ,

$$\Phi(\beta s) = C_{2/q,q'} \left( \frac{1}{\beta} \left( \frac{F}{s} \cap B_\beta \right) \right) \approx C_{2/q,q'} \left( \frac{F}{s} \cap B_\beta \right) \geq c_1 \Phi(s).$$

If  $\alpha \in [1/2, 1]$ ,

$$\Phi(\alpha s) = C_{2/q,q'} \left( \frac{1}{\alpha} \left( \frac{F}{s} \cap B_\alpha \right) \right) \approx C_{2/q,q'} \left( \frac{F}{s} \cap B_\alpha \right) \leq c_2 \Phi(s).$$

*Step 2.* By (2.92)

$$C_{2/q,q'} \left( \frac{F}{d_{n+1}} \cap B_1 \right) \leq c_2 C_{2/q,q'} \left( \frac{F}{s} \cap B_1 \right) \quad \forall s \in [d_{n+1}, d_{n+2}],$$

and  $n \leq a_t$ . Then

$$\begin{aligned} c_2 \int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-s^2/4t} C_{2/q,q'} \left( \frac{F}{s} \cap B_1 \right) s \, ds \\ \geq C_{2/q,q'} \left( \frac{F}{d_{n+1}} \cap B_1 \right) \int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-s^2/4t} s \, ds. \end{aligned}$$

Using the fact that  $N - \frac{2}{q-1} \geq 0$ , we get,

$$\int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} s \, ds \geq e^{-\frac{d_{n+2}^2}{4t}} d_{n+1}^{N-\frac{2}{q-1}+1} (d_{n+2} - d_{n+1}) \quad (2.93)$$

$$\geq \frac{t}{4e^2} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{d_{n+1}^2}{4t}}. \quad (2.94)$$

Thus

$$\bar{u}_F(x, t) \leq \frac{C}{t^{1+\frac{N}{2}}} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'} \left( \frac{1}{s} F \cap B_1 \right) s \, ds, \quad (2.95)$$

which ends the proof.  $\square$

### 3 Estimate from below

If  $\mu \in \mathfrak{M}_+^q(\mathbb{R}^N) \cap \mathfrak{M}^b(\mathbb{R}^N)$ , we denote by  $u_\mu = u_{\mu,0}$  the solution of

$$\begin{cases} \partial_t u_\mu - \Delta u_\mu + u_\mu^q = 0 & \text{in } Q_T, \\ u_\mu(\cdot, 0) = \mu & \text{in } \mathbb{R}^N. \end{cases} \quad (3.1)$$

The maximal  $\sigma$ -moderate solution of (1.1) which has an initial trace vanishing outside a closed set  $F$  is defined by

$$\underline{u}_F = \sup \left\{ u_\mu : \mu \in \mathfrak{M}_+^q(\mathbb{R}^N) \cap \mathfrak{M}^b(\mathbb{R}^N), \mu(F^c) = 0 \right\}. \quad (3.2)$$

The main result of this section is the next one

**Theorem 3.1** *Assume  $q \geq q_c$ . There exists a constant  $C_2 = C_2(N, q, T) > 0$  such that, for any closed subset  $F \subset \mathbb{R}^N$ , there holds*

$$\underline{u}_F(x, t) \geq C_2 W_F(x, t) \quad \forall (x, t) \in Q_T. \quad (3.3)$$

We first assume that  $F$  is compact, and we denote it by  $K$ . The first observation is that if  $\mu \in \mathfrak{M}_+^q(\mathbb{R}^N)$ ,  $u_\mu \in L^q(Q_T)$  (see lemma below) and  $0 \leq u_\mu \leq \mathbb{H}[\mu] := \mathbb{H}_\mu$ . Therefore

$$u_\mu \geq \mathbb{H}[\mu] - \mathbb{G}[\mathbb{H}[\mu]^q], \quad (3.4)$$

where  $\mathbb{G}$  is the parabolic Green potential in  $Q_T$  defined by

$$\mathbb{G}[f](t) = \int_0^t \mathbb{H}[f(s)](t-s) ds = \int_0^t \int_{\mathbb{R}^N} H(\cdot, y, t-s) f(y, s) dy ds.$$

The main idea of the proof is as follows. For any  $(x, t) \in Q_T$ , construct a measure  $\mu = \mu(x, t) \in \mathfrak{M}_+^q(\mathbb{R}^N)$  such that there holds

$$\mathbb{H}[\mu](x, t) \geq C W_K(x, t) \quad \forall (x, t) \in Q_T, \quad (3.5)$$

and

$$\mathbb{G}(\mathbb{H}[\mu])^q \leq C \mathbb{H}[\mu] \quad \text{in } Q_T, \quad (3.6)$$

with constants  $C$  depends only on  $N, q$ , and  $T$ . Then replace  $\mu$  by  $\mu_\epsilon = \epsilon \mu$  with  $\epsilon = (2C)^{-\frac{1}{q-1}}$  in order to derive

$$u_{\mu_\epsilon} \geq 2^{-1} \mathbb{H}_{\mu_\epsilon} \geq 2^{-1} C W_K. \quad (3.7)$$

From this follows

$$\underline{u}_K \geq 2^{-1} \mathbb{H}_{\mu_\epsilon} \geq 2^{-1} C W_K. \quad (3.8)$$

and the proof of Theorem 3.1 with  $C_2 = 2^{-1}C$ . In the following sections we describe the construction of measures  $\mu(x, t)$  satisfying (3.5) and (3.6).

### 3.1 Estimate from below of the solution of the heat equation

The purely spatial slicing used is the trace on  $\mathbb{R}^N \times \{0\}$  of an *extended slicing* in  $Q_T$  which is constructed as follows: if  $K$  is a compact subset of  $\mathbb{R}^N$ ,  $m = (x, t)$ , we define  $d_K$ ,  $\lambda$ ,  $d_n$  and  $a_t$  as in Section 2.3. Let  $\alpha \in (0, 1)$  to be fixed later on, we define  $\mathcal{T}_n$  for  $n \in \mathbb{Z}$  by

$$\mathcal{T}_n = \begin{cases} \mathcal{B}_{\sqrt{t(n+1)}}^2(m) \setminus \mathcal{B}_{\sqrt{tn}}^2(m) & \text{if } n \geq 1, \\ \mathcal{B}_{\alpha^{-n}\sqrt{t}}^2(m) \setminus \mathcal{B}_{\alpha^{1-n}\sqrt{t}}^2(m) & \text{if } n \leq 0, \end{cases}$$

and put

$$\mathcal{T}_n^* = \mathcal{T}_n \cap \{s : 0 \leq s \leq t\}, \quad \text{for } n \in \mathbb{Z}.$$

We recall that for  $n \in \mathbb{N}_*$ ,

$$\mathcal{Q}_n = K \cap \mathcal{B}_{\sqrt{t(n+1)}}^2(m) = K \cap B_{d_n}(x)$$

and

$$K_n = K \cap \mathcal{T}_{n+1} = K \cap (B_{d_{n+1}}(x) \setminus B_{d_n}(x)).$$

Let  $\nu_n \in \mathfrak{M}_+^b(\mathbb{R}^N) \cap W^{-2/q, q}(\mathbb{R}^N)$  be the  $q$ -capacitary measure of the set  $K_n/d_{n+1}$ . See [1, Sec. 2.2]. Such a measure has support in  $K_n/d_{n+1}$  and

$$\nu_n(K_n/d_{n+1}) = C_{2/q, q'}(K_n/d_{n+1}) \quad \text{and} \quad \|\nu_n\|_{W^{-2/q, q'}(\mathbb{R}^N)} = (C_{2/q, q'}(K_n/d_{n+1}))^{1/q}. \quad (3.9)$$

We define  $\mu_n$  as follows

$$\mu_n(A) = d_{n+1}^{N - \frac{2}{q-1}} \nu_n(A/d_{n+1}) \quad \forall A \subset K_n, \quad A \text{ Borel}, \quad (3.10)$$

and set

$$\mu_{t, K} = \sum_{n=0}^{a_t} \mu_n,$$

and

$$\mathbb{H}_{\mu_{t, K}} = \sum_{n=0}^{a_t} \mathbb{H}_{\mu_n}. \quad (3.11)$$

**Proposition 3.2** *Let  $q \geq q_c$ , then there holds*

$$\mathbb{H}_{\mu_{t, K}}(x, t) \geq \frac{1}{(4\pi t)^{\frac{N}{2}}} \sum_{n=0}^{a_t} e^{-\frac{n+1}{4}} d_{n+1}^{N - \frac{2}{q-1}} C_{2/q, q'} \left( \frac{K_n}{d_{n+1}} \right), \quad (3.12)$$

in  $\mathbb{R}^N \times (0, T)$ .

*Proof.* Since

$$\mathbb{H}_{\mu_n}(x, t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{K_n} e^{-\frac{|x-y|^2}{4t}} d\mu_n, \quad (3.13)$$

and

$$y \in K_n \implies |x - y| \leq d_{n+1},$$

(3.12) follows because of (3.10) and (3.11).  $\square$

### 3.2 Estimate from above of the nonlinear term

We write (3.4) under the form

$$\begin{aligned} u_\mu(x, t) &\geq \sum_{n \in \mathbb{Z}} \mathbb{H}_{\mu_n}(x, t) - \int_0^t \int_{\mathbb{R}^N} H(x, y, t-s) \left[ \sum_{n \in A_K} \mathbb{H}_{\mu_n}(y, s) \right]^q dy ds \\ &= I_1 - I_2. \end{aligned} \quad (3.14)$$

since  $\mu_n = 0$  if  $n \notin A_K = \mathbb{N} \cap [1, a_t]$ , and

$$\begin{aligned} I_2 &= \frac{1}{(4\pi)^{\frac{N}{2}}} \int_0^t \int_{\mathbb{R}^N} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[ \sum_{n \in A_K} \mathbb{H}_{\mu_n}(y, s) \right]^q dy ds \\ &= \frac{1}{(4\pi)^{\frac{N}{2}}} (J_\ell + J'_\ell), \end{aligned} \quad (3.15)$$

for some  $\ell \in \mathbb{N}^*$  to be fixed later on, where

$$J_\ell = \sum_{p \in \mathbb{Z}} \int \int_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[ \sum_{n < p+\ell} \mathbb{H}_{\mu_n}(y, s) \right]^q dy ds,$$

and

$$J'_\ell = \sum_{p \in \mathbb{Z}} \int \int_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[ \sum_{n \geq p+\ell} \mathbb{H}_{\mu_n}(y, s) \right]^q dy ds.$$

The next estimate will be used several times in the sequel.

**Lemma 3.3** *Let  $0 < a < b$  and  $t > 0$ , then,*

$$\max \left\{ \sigma^{-\frac{N}{2}} e^{-\frac{\rho^2}{4\sigma}} : 0 \leq \sigma \leq t, at \leq \rho^2 + \sigma \leq bt \right\} = e^{\frac{1}{4}} \begin{cases} t^{-\frac{N}{2}} e^{-\frac{a}{4}} & \text{if } \frac{a}{2N} > 1, \\ \left( \frac{2N}{at} \right)^{\frac{N}{2}} e^{-\frac{N}{2}} & \text{if } \frac{a}{2N} \leq 1. \end{cases}$$

*Proof.* Set

$$\mathcal{J}(\rho, \sigma) = \sigma^{-\frac{N}{2}} e^{-\frac{\rho^2}{4\sigma}}$$

and

$$\mathcal{K}_{a,b,t} = \{(\rho, \sigma) \in [0, \infty) \times (0, t] : at \leq \rho^2 + \sigma \leq bt\}.$$

We first notice that, for fixed  $\sigma$ , the maximum of  $\mathcal{J}(\cdot, \sigma)$  is achieved for  $\rho$  minimal. If  $\sigma \in [at, bt]$  the minimal value of  $\rho$  is 0, while if  $\sigma \in (0, at)$ , the minimum of  $\rho$  is  $\sqrt{at - \sigma}$ .

- Assume first  $a \geq 1$ , then  $\mathcal{J}(\sqrt{at - \sigma}, \sigma) = e^{\frac{1}{4}} \sigma^{-\frac{N}{4}} e^{-\frac{at}{4\sigma}}$ . Thus if  $1 \leq a/2N$ , the minimal value of  $\mathcal{J}(\sqrt{at - \sigma}, \sigma)$  is  $e^{\frac{1-2N}{4}} \left( \frac{2N}{at} \right)^{\frac{N}{2}}$ , while if  $a/2N < 1 \leq a$ , the minimum is  $e^{\frac{1}{4}} t^{-\frac{N}{2}} e^{-\frac{a}{4}}$ .

- Assume now  $a \leq 1$ . Then

$$\begin{aligned} \max\{\mathcal{J}(\rho, \sigma) : (\rho, \sigma) \in \mathcal{K}_{a,b,t}\} &= \max\left\{\max_{\sigma \in (at, t]} \mathcal{J}(0, \sigma), \max_{\sigma \in (0, at]} \mathcal{J}(\sqrt{at - \sigma}, \sigma)\right\} \\ &= \max\left\{(at)^{-\frac{N}{2}}, e^{\frac{1-2N}{4}} \left(\frac{2N}{at}\right)^{\frac{N}{2}}\right\} \\ &= e^{\frac{1-2N}{4}} \left(\frac{2N}{at}\right)^{\frac{N}{2}}. \end{aligned}$$

Combining these two estimates, we derive the result.  $\square$

*Remark.* The following variant of Lemma 3.3 will be useful in the sequel: *For any  $\theta \geq 1/2N$  there holds*

$$\max\{\mathcal{J}(\rho, \sigma) : (\rho, \sigma) \in \mathcal{K}(a, b, t)\} \leq e^{\frac{1}{4}} \left(\frac{2N\theta}{t}\right)^{\frac{N}{2}} e^{-\frac{a}{4}} \quad \text{if } \theta a \geq 1. \quad (3.16)$$

**Lemma 3.4** *There exists a positive constant  $C = C(N, \ell, q)$  such that*

$$J_\ell \leq Ct^{-\frac{N}{2}} \sum_{n=1}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-(1+(n-\ell)_+)/4} C_{2/q, q'} \left(\frac{K_n}{d_{n+1}}\right). \quad (3.17)$$

*Proof.* The set of the  $p$ 's for the summation in  $J_\ell$  is reduced to  $\mathbb{Z} \cap [-\ell + 2, \infty)$ , thus we write

$$J_\ell = J_{1,\ell} + J_{2,\ell}$$

where

$$J_{1,\ell} = \sum_{p=2-\ell}^0 \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[ \sum_{n < p+\ell} \mathbb{H}_{\mu_n}(y, s) \right]^q$$

and

$$J_{2,\ell} = \sum_{p=1}^{\infty} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[ \sum_{n < p+\ell} \mathbb{H}_{\mu_n}(y, s) \right]^q.$$

If  $p = 2 - \ell, \dots, 0$ ,

$$(y, s) \in \mathcal{T}_p^* \implies t\alpha^{2-2p} \leq |x-y|^2 + t-s \leq t\alpha^{-2p},$$

and, if  $p \geq 1$

$$(y, s) \in \mathcal{T}_p^* \implies pt \leq |x-y|^2 + t-s \leq (p+1)t.$$

By Lemma 3.3 and (3.16), there exists  $C = C(N, \ell, \alpha) > 0$  such that

$$\max\left\{(t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} : (y, s) \in \mathcal{T}_p^*\right\} \leq Ct^{-\frac{N}{2}} e^{-\alpha^{2-2p}/4}, \quad (3.18)$$

if  $p = 2 - \ell, \dots, 0$ , and

$$\max \left\{ (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} : (y, s) \in \mathcal{T}_p^* \right\} \leq C t^{-\frac{N}{2}} e^{-p/4}, \quad (3.19)$$

if  $p \geq 1$ . When  $p = 2 - \ell, \dots, 0$

$$\left[ \sum_1^{p+\ell-1} \mathbb{H}_{\mu_n}(y, s) \right]^q \leq C \sum_1^{p+\ell-1} \mathbb{H}_{\mu_n}^q(y, s), \quad (3.20)$$

for some  $C = C(\ell, q) > 0$ , thus

$$\begin{aligned} J_{1,\ell} &\leq C t^{-\frac{N}{2}} \sum_{p=2-\ell}^0 e^{-\frac{\alpha^2-2p}{4}} \sum_{n=1}^{p+\ell-1} \|\mathbb{H}_{\mu_n}\|_{L^q(Q_t)}^q \\ &\leq C t^{-\frac{N}{2}} \sum_{n=1}^{\ell-1} \|\mathbb{H}_{\mu_n}\|_{L^q(Q_t)}^q \sum_{p=n-\ell+1}^0 e^{-\frac{\alpha^2-2p}{4}} \\ &\leq C t^{-\frac{N}{2}} e^{-\frac{\alpha^2-2\ell-2}{4}} \sum_{n=1}^{\ell-1} \|\mathbb{H}_{\mu_n}\|_{L^q(Q_t)}^q. \end{aligned} \quad (3.21)$$

If the set of  $p$ 's is not upper bounded, we introduce some parameter  $\delta > 0$  to be made precise later on. Then

$$\left[ \sum_1^{p+\ell-1} \mathbb{H}_{\mu_n}(y, s) \right]^q \leq \left[ \sum_1^{p+\ell-1} e^{\delta q' \frac{n}{4}} \right]^{q/q'} \sum_1^{p+\ell-1} e^{-\frac{\delta q n}{4}} \mathbb{H}_{\mu_n}^q(y, s), \quad (3.22)$$

with  $q' = q/(q-1)$ . If, by convention  $\mu_n = 0$  whenever  $n > a_t$ , we obtain, for some  $C > 0$  which depends also on  $\delta$ ,

$$\begin{aligned} J_{2,\ell} &\leq C t^{-\frac{N}{2}} \sum_{p=1}^{\infty} e^{\frac{\delta(p+\ell-1)q-p}{4}} \sum_{n=1}^{p+\ell-1} e^{-\frac{\delta q n}{4}} \|\mathbb{H}_{\mu_n}\|_{L^q(Q_t)}^q \\ &\leq C t^{-\frac{N}{2}} \sum_{n=1}^{\infty} \|\mathbb{H}_{\mu_n}\|_{L^q(Q_t)}^q e^{-\frac{\delta q n}{4}} \sum_{p=(n-\ell+1) \vee 1}^{\infty} e^{\frac{\delta(p+\ell-1)q-p}{4}} \\ &\leq C t^{-\frac{N}{2}} \sum_{n=1}^{\infty} e^{-\frac{1+(n-\ell)+}{4}} \|\mathbb{H}_{\mu_n}\|_{L^q(Q_t)}^q. \end{aligned} \quad (3.23)$$

Notice that we choose  $\delta$  such that  $\delta \ell q < 1$ . Combining (3.21) and (3.23), we derive (3.17) from Lemma 2.11, (3.9) and (3.10).  $\square$

The set of indices  $p$  for which the  $\mu_n$  terms are not zero in  $J'_\ell$  is  $\mathbb{Z} \cap (-\infty, a_t - \ell]$ . We write

$$J'_\ell = J'_{1,\ell} + J'_{2,\ell},$$

where

$$J'_{1,\ell} = \sum_{p=-\infty}^0 \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[ \sum_{n=1 \vee p+\ell}^{\infty} \mathbb{H}_{\mu_n}(y, s) \right]^q dy ds,$$

and

$$J'_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[ \sum_{n=p+\ell}^{\infty} \mathbb{H}_{\mu_n}(y, s) \right]^q dy ds.$$

**Lemma 3.5** *There exists a constant  $C = C(N, q, \ell) > 0$  such that*

$$J'_{1,\ell} \leq C t^{1-\frac{Nq}{2}} \sum_{n=0}^{a_t} e^{-\frac{(1+\beta_0)(n-h)_+}{4}} d_{n+1}^{Nq-2q'} C_{2/q, q'}^q \left( \frac{K_n}{d_{n+1}} \right), \quad (3.24)$$

where  $\beta_0 = (q-1)/4$  and  $h = 2q(q+1)/(q-1)^2$ .

*Proof.* Since

$$(y, s) \in \mathcal{T}_p^*, \text{ and } (z, 0) \in K_n \implies |y-z| \geq (\sqrt{n} - \alpha^{-p})\sqrt{t}, \quad (3.25)$$

there holds

$$\mathbb{H}_{\mu_n}(y, s) \leq (4\pi s)^{-\frac{N}{2}} e^{-\frac{(\sqrt{n}-\alpha^{-p})^2 t}{4s}} \mu_n(K_n) \leq C t^{-\frac{N}{2}} e^{-\frac{(\sqrt{n}-\alpha^{-p})^2}{4}} \mu_n(K_n),$$

by Lemma 3.3. Let  $\{\epsilon_n\}$  be a sequence of positive numbers such that

$$A_\epsilon = \sum_{n=1}^{\infty} \epsilon_n^{q'} < \infty,$$

then

$$\begin{aligned} J'_{1,\ell} &\leq C A_\epsilon^{q/q'} t^{-\frac{Nq}{2}} \sum_{p=-\infty}^0 \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \sum_{n=1 \vee (p+\ell)}^{\infty} \epsilon_n^{-q} e^{-q \frac{(\sqrt{n}-\alpha^{-p})^2}{4}} \mu_n^q(K_n) ds dy \\ &\leq C A_\epsilon^{q/q'} t^{-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) \sum_{p=0 \wedge (n-\ell)}^{\infty} e^{-q \frac{(\sqrt{n}-\alpha^{-p})^2}{4}} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} ds dy \\ &\leq C A_\epsilon^{q/q'} t^{-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) e^{-\frac{q(\sqrt{n}-1)^2}{4}} \iint_{\{\cup_{p \leq 0} \mathcal{T}_p^*\}} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} ds dy \\ &\leq C A_\epsilon^{q/q'} t^{1-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) e^{-\frac{q(\sqrt{n}-1)^2}{4}}. \end{aligned} \quad (3.26)$$

Set  $h = 2q(q+1)/(q-1)^2$  and  $Q = (1+q)/2$ , then  $q(\sqrt{n}-1)^2 \geq Q(n-h)_+$  for any  $n \geq 1$ . If we choose  $\epsilon_n = e^{-\frac{(q-1)(n-h)_+}{16q}}$ , there holds  $\epsilon_n^{-q} e^{-\frac{q(\sqrt{n}-1)^2}{4}} \leq e^{-\frac{(q+3)(n-h)_+}{16}}$ . Finally

$$J'_{1,\ell} \leq C t^{1-\frac{Nq}{2}} \sum_{n=1}^{\infty} e^{-\frac{(1+\beta_0)(n-h)_+}{4}} \mu_n^q(K_n),$$



with  $\beta_0 = (q-1)/4$ , which yields to (3.24) by the choice of the  $\mu_n$ .  $\square$

In order to make easier the obtention of the estimate of the term  $J'_{2,\ell}$ , we first give the proof in dimension 1.

**Lemma 3.6** *Assume  $N = 1$  and  $\ell$  is an integer larger than 1. There exists a positive constant  $C = C(q, \ell) > 0$  such that*

$$J'_{2,\ell} \leq Ct^{-1/2} \sum_{n=\ell}^{at} e^{-\frac{n}{4}} d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'} \left( \frac{K_n}{d_{n+1}} \right). \quad (3.27)$$

*Proof.* If  $(y, s) \in \mathcal{T}_p^*$  and  $z \in K_n$  ( $p \geq 1$ ,  $n \geq p = \ell$ ), there holds  $|x - y| \geq \sqrt{t}\sqrt{p}$  and  $|y - z| \geq \sqrt{t}(\sqrt{n} - \sqrt{p+1})$ . Therefore

$$J'_{2,\ell} \leq C\sqrt{t} \sum_{p=1}^{at} \frac{1}{\sqrt{p}} \int_0^t e^{-\frac{pt}{4(t-s)}} \left( \sum_{n=p+\ell}^{at} s^{-1/2} e^{-\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n(K_n) \right)^q.$$

If  $\epsilon \in (0, q)$  is some positive parameter which will be made more precise later on, there holds

$$\begin{aligned} & \left( \sum_{n=p+\ell}^{at} s^{-1/2} e^{-\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n(K_n) \right)^q \\ & \leq \left( \sum_{n=p+\ell}^{at} e^{-\epsilon q' \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \right)^{q/q'} \sum_{n=p+\ell}^{at} s^{-\frac{q}{2}} e^{-(q-\epsilon) \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n^q(K_n), \end{aligned}$$

by Hölder's inequality. By comparison between series and integrals and using Gauss integral

$$\begin{aligned} \sum_{n=p+\ell}^{at} e^{-\epsilon q' \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} & \leq \int_{p+\ell}^{\infty} e^{-\epsilon q' \frac{(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx \\ & = 2 \int_{\sqrt{p+\ell}-\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' x^2 t}{4s}} (x + \sqrt{p+1}) dx \\ & \leq \frac{4s}{\epsilon q' t} e^{-\epsilon q' \frac{(\sqrt{p+\ell}-\sqrt{p+1})^2 t}{4s}} + 2\sqrt{p+1} \int_{\sqrt{p+\ell}-\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' x^2 t}{4s}} dx \\ & \leq C \sqrt{\frac{(p+1)s}{t}} e^{-\epsilon q' \frac{(\sqrt{p+\ell}-\sqrt{p+1})^2 t}{4s}} \\ & \leq C \sqrt{\frac{(p+1)s}{t}}. \end{aligned}$$

If we set  $q_\epsilon = q - \epsilon$ , then

$$J'_{2,\ell} \leq C \epsilon^{-q'/q} t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{\infty} \mu_n^q(K_n) \sum_{p=1}^{n-\ell} p^{\frac{q-2}{2}} \int_0^t (t-s)^{-1/2} s^{-1/2} e^{-\frac{pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds.$$

where  $C = C(\epsilon, q) > 0$ . Since

$$\begin{aligned} \int_0^t (t-s)^{-1/2} s^{-1/2} e^{-\frac{pt}{4(t-s)}} e^{-q\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds \\ = \int_0^1 (1-s)^{-1/2} s^{-1/2} e^{-\frac{p}{4(1-s)}} e^{-q\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2}{4s}} ds, \end{aligned}$$

we can apply Lemma A.1 with  $a = 1/2$ ,  $b = 1/2$ ,  $A = \sqrt{p}$  and  $B = \sqrt{q\epsilon}(\sqrt{n} - \sqrt{p+1})$ . In this range of indices  $B \geq \sqrt{q\epsilon}(\sqrt{p+\ell} - \sqrt{p+1}) \geq \frac{\sqrt{q\epsilon}(\ell-1)}{\sqrt{p}}$ , thus  $\kappa = \sqrt{q\epsilon}(\ell-1)$  and

$$\sqrt{\frac{A}{A+B}} \sqrt{\frac{B}{A+B}} \leq p^{\frac{1}{4}} n^{-1/2} (\sqrt{n} - \sqrt{p})^{1/2}.$$

Therefore

$$\int_0^t (t-s)^{-1/2} s^{-\frac{q}{2}} e^{-\frac{pt}{4(t-s)}} e^{-q \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds \leq \frac{C p^{\frac{1}{4}} (\sqrt{n} - \sqrt{p})^{1/2}}{\sqrt{n}} e^{-\frac{(\sqrt{p} + \sqrt{q\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}, \quad (3.28)$$

which implies

$$J'_{2,\ell} \leq C t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{a_t} \frac{\mu_n^q(K_n)}{\sqrt{n}} \sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} (\sqrt{n} - \sqrt{p})^{1/2} e^{-\frac{(\sqrt{p} + \sqrt{q\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}, \quad (3.29)$$

where  $C$  depends of  $\epsilon$ ,  $q$  and  $\ell$ . By Lemma A.2

$$J'_{2,\ell} \leq C t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{a_t} n^{\frac{q-3}{2}} e^{-\frac{n}{4}} \mu_n^q(K_n) \quad (3.30)$$

Because  $\mu_n(K_n) = d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'} \left( \frac{K_n}{d_{n+1}} \right)$  (remember  $N = 1$ ) and  $\text{diam} \frac{K_n}{d_{n+1}} \leq n^{-1}$ , there holds

$$\mu_n^q(K_n) \leq C \left( \frac{\sqrt{t}}{\sqrt{n}} \right)^{q-3} \mu_n(K_n) = C \left( \frac{\sqrt{t}}{\sqrt{n}} \right)^{q-3} d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'} (K_n/d_{n+1}) \quad (3.31)$$

and inequality (3.27) follows.  $\square$

Next we give the general proof. For this task we will use again the quasi-additivity with separated partitions.

**Lemma 3.7** *Assume  $N \geq 2$  and  $\ell$  is an integer larger than 1. There exists a positive constant  $C_1 = C_1(q, N, \ell) > 0$  such that*

$$J'_{2,\ell} \leq C_1 t^{-\frac{N}{2}} \sum_{n=\ell}^{a_t} e^{-\frac{n}{4}} d_{n+1}^{N-\frac{2}{q-1}} C_{2/q,q'} \left( \frac{K_n}{d_{n+1}} \right). \quad (3.32)$$

*Proof.* As in the proof of Theorem 2.22, we know that there exists a finite number  $J$ , depending only on the dimension  $N$ , of separated sub-partitions  $\{\#\Theta_{t,n}^h\}_{h=1}^J$  of the rescaled sets  $\tilde{T}_n = \sqrt{\frac{n+1}{t}} T_n$  by the  $N$ -dim balls  $B_2(\tilde{a}_{n,j})$  where  $\tilde{a}_{n,j} = \sqrt{\frac{n+1}{t}} a_{n,j}$ ,  $|a_{n,j}| = \frac{d_{n+1} + d_n}{2}$  and  $|a_{n,j} - a_{n,k}| \geq \sqrt{\frac{4t}{n+1}}$ . Furthermore  $\#\Theta_{t,n}^h \leq Cn^{N-1}$ . We denote  $K_{n,j} = K_n \cap B_{\sqrt{\frac{t}{n+1}}}(a_{n,j})$ . We write  $\mu_n = \sum_{h=1}^J \mu_n^h$ , and accordingly  $J'_{2,\ell} = \sum_{h=1}^J J'_{2,\ell}^h$ , where  $\mu_n^h = \sum_{j \in \Theta_{t,n}^h} \mu_{n,j}$ , and  $\mu_{n,j}$  are the capacitary measures of  $K_{n,j}$  relative to  $B_{n,j} = B_{6t/5\sqrt{n}}(a_{n,j})$ , which means

$$\nu_{n,j}(K_{n,j}) = C_{2/q,q'}^{B_{n,j}}(K_{n,j}) \quad \text{and} \quad \|\nu_{n,j}\|_{W^{-2/q,q'}(B_{n,j})} = \left(C_{2/q,q'}^{B_{n,j}}(K_{n,j})\right)^{1/q}. \quad (3.33)$$

Thus

$$J'_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[ \sum_{n=p+\ell}^{\infty} \sum_{h=1}^J \sum_{j \in \Theta_{t,n}^h} \mathbb{H}_{\mu_{n,j}}(y,s) \right]^q dy ds.$$

We denote

$$J'_{2,\ell}^h = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[ \sum_{n=p+\ell}^{\infty} \sum_{j \in \Theta_{t,n}^h} \mathbb{H}_{\mu_{n,j}}(y,s) \right]^q dy ds,$$

and clearly

$$J'_{2,\ell} \leq C \sum_{h=1}^J J'_{2,\ell}^h, \quad (3.34)$$

where  $C$  depends only on  $N$  and  $q$ . For integers  $n$  and  $p$  such that  $n \geq \ell + 1$ , we set

$$\lambda_{n,j,y} = \inf\{|y-z| : z \in B_{\sqrt{t}/\sqrt{n+1}}(a_{n,j})\} = |y - a_{n,j}| - \frac{\sqrt{t}}{\sqrt{n+1}}.$$

Therefore

$$\begin{aligned} \sum_{n=p+\ell}^{a_t} \int_{K_n} e^{-\frac{|y-z|^2}{4s}} d\mu_n^h(z) &= \sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} \int_{K_{n,j}} e^{-\frac{|y-z|^2}{4s}} d\mu_{n,j}(z) \\ &\leq \left( \sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} e^{-\epsilon q' \frac{\lambda_{n,j,y}^2}{4s}} \right)^{1/q'} \left( \sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} e^{-q \lambda_{n,j,y}^2 \frac{1-\epsilon}{4s}} \mu_{n,j}^q(K_{n,j}) \right)^{1/q} \end{aligned}$$

where  $\epsilon > 0$  will be made precise later on.

*Step 1* We claim that

$$\sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} e^{-\epsilon q' \frac{\lambda_{n,j,y}^2}{4s}} \leq C \sqrt{\frac{ps}{t}} \quad (3.35)$$

where  $C$  depends on  $\epsilon$ ,  $q$  and  $N$ . If  $y$  is fixed in  $T_p$ , we denote by  $z_y$  the point of  $T_n$  which solves  $|y - z_y| = \text{dist}(y, T_n)$ . Thus

$$\sqrt{t}(\sqrt{n} - \sqrt{p+1}) \leq |y - z_y| \leq t(\sqrt{n} - \sqrt{p}).$$

Let  $Y = y\sqrt{t(p+1)}/|y|$ . On the axis  $\overrightarrow{0Y}$  we set  $\mathbf{e} = Y/|Y|$ , consider the points  $b_k = (k\sqrt{t}/\sqrt{n})\mathbf{e}$  where  $-n \leq k \leq n$  and denote by  $G_{n,k}$  the spherical shell obtained by intersecting the spherical shell  $T_n$  with the domain  $H_{n,k}$  which is the set of points in  $\mathbb{R}^N$  limited by the hyperplanes orthogonal to  $\overrightarrow{0Y}$  going through  $((k+1)\sqrt{t}/\sqrt{n})\mathbf{e}$  and  $((k-1)\sqrt{t}/\sqrt{n})\mathbf{e}$ . The number of points  $a_{n,j} \in G_{n,k}$  is smaller than  $C(n+1-|k|)^{N-2}$ , where  $C$  depends only on  $N$ , and we denote by  $\Lambda_{n,k}$  the set of  $j \in \Theta_{t,n}$  such that  $a_{n,j} \in G_{n,k}$ . Furthermore, if  $a_{n,j} \in G_{n,k}$  elementary geometric considerations (Pythagora's theorem) imply that  $\lambda_{n,j,y}^2$  is greater than  $t(n+p+1-2k\sqrt{p+1}/\sqrt{n})$ . Therefore

$$\sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}} e^{-\epsilon q' \frac{\lambda_{n,j,y}^2}{4s}} \leq C \sum_{n=p+\ell}^{a_t} \sum_{k=-n}^n (n+1-|k|)^{N-2} e^{-\frac{\epsilon q' (n+p+1-2k\sqrt{p+1}/\sqrt{n})t}{4s\sqrt{n}}}. \quad (3.36)$$

Case  $N = 2$ . Summing a geometric series and using the inequality  $\frac{e^u}{e^u-1} \leq 1 + u^{-1}$  for  $u > 0$ , we obtain

$$\begin{aligned} \sum_{k=-n}^n e^{\frac{\epsilon q' (k\sqrt{p+1})t}{2s\sqrt{n}}} &\leq e^{\frac{\epsilon q' t\sqrt{n(p+1)}}{2s}} \frac{e^{\frac{\epsilon q' t\sqrt{p+1}}{2s\sqrt{n}}}}{e^{\frac{\epsilon q' t\sqrt{p+1}}{2s\sqrt{n}}}-1} \\ &\leq e^{\frac{\epsilon q' t\sqrt{n(p+1)}}{2s}} \left(1 + \frac{2s\sqrt{n}}{\epsilon q' t\sqrt{p+1}}\right). \end{aligned} \quad (3.37)$$

Thus, by comparison between series and integrals,

$$\begin{aligned} \sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}} e^{-\frac{\epsilon q' \lambda_{n,j,y}^2}{4s}} &\leq C \sum_{n=p+\ell}^{a_t} \left(1 + \frac{s\sqrt{n}}{t\sqrt{p}}\right) e^{-\frac{\epsilon q' (\sqrt{n}-\sqrt{p+1})^2}{4s}} \\ &\leq C \int_{p+1}^{\infty} e^{-\frac{\epsilon q' (\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx \\ &\quad + \frac{Cs}{t\sqrt{p}} \int_{p+1}^{\infty} \sqrt{x} e^{-\frac{\epsilon q' (\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx. \end{aligned} \quad (3.38)$$

Next

$$\begin{aligned} \int_{p+1}^{\infty} e^{-\frac{\epsilon q' (\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx &= 2 \int_{\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' (y-\sqrt{p+1})^2 t}{4s}} y dy \\ &= 2 \int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} y dy + 2\sqrt{p+1} \int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} dy \\ &= \frac{2s}{t} \int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} z dz + 2\sqrt{\frac{(p+1)s}{t}} \int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} dz, \end{aligned} \quad (3.39)$$

and

$$\begin{aligned}
\int_{p+1}^{\infty} \sqrt{x} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx &= 2 \int_{\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q'(y-\sqrt{p+1})^2 t}{4s}} y^2 dy \\
&= 2 \int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} (y + \sqrt{p+1})^2 dy \\
&\leq 4 \int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} y^2 dy + 4(p+1) \int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} dy \\
&\leq 4 \left(\frac{s}{t}\right)^{3/2} \int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} z^2 dz + 4(p+1) \sqrt{\frac{s}{t}} \int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} dz
\end{aligned} \tag{3.40}$$

Jointly with (3.38), these inequalities imply

$$\sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}} e^{-\frac{\epsilon q' \lambda_{n,j,y}^2}{4s}} \leq C \sqrt{\frac{ps}{t}}. \tag{3.41}$$

*Case  $N > 2$ .* Because the value of the right-hand side of (3.36) is an increasing value of  $N$ , it is sufficient to prove (3.35) when  $N$  is even, say  $(N-2)/2 = d \in \mathbb{N}_*$ . There holds

$$\sum_{k=-n}^n (n+1-|k|)^d e^{\frac{\epsilon q'(k\sqrt{p+1})t}{2s\sqrt{n}}} \leq 2 \sum_{k=0}^n (n+1-k)^d e^{\frac{\epsilon q'(k\sqrt{p+1})t}{2s\sqrt{n}}}. \tag{3.42}$$

We set

$$\alpha = \epsilon q' \frac{t\sqrt{p+1}}{2s\sqrt{n}} \quad \text{and} \quad I_d = \sum_{k=0}^n (n+1-k)^d e^{k\alpha}.$$

Since

$$e^{k\alpha} = \frac{e^{(k+1)\alpha} - e^{k\alpha}}{e^{\alpha} - 1},$$

we use Abel's transform to obtain

$$\begin{aligned}
I_d &= \frac{1}{e^{\alpha} - 1} \left( e^{(n+1)\alpha} - (n+1)^d + \sum_{k=1}^n ((n+2-k)^d - (n+1-k)^d) e^{k\alpha} \right) \\
&\leq \frac{1}{e^{\alpha} - 1} \left( (1-d)e^{(n+1)\alpha} - (n+1)^d + de^{\alpha} \sum_{k=1}^n ((n+1-k)^{d-1}) e^{k\alpha} \right).
\end{aligned}$$

Therefore the following induction holds

$$I_d \leq \frac{de^{\alpha}}{e^{\alpha} - 1} I_{d-1}. \tag{3.43}$$

In (3.37), we have already used the fact that

$$\frac{de^{\alpha}}{e^{\alpha} - 1} \leq C \left( 1 + \frac{s\sqrt{n}}{t\sqrt{p}} \right),$$

and

$$I_d \leq C \left( 1 + \left( \frac{s\sqrt{n}}{t\sqrt{p}} \right)^{d+1} \right) I_0.$$

Thus (3.38) is replaced by

$$\begin{aligned} \sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}} e^{-\frac{\epsilon q' \lambda_{n,j,y}^2}{4s}} &\leq C \sum_{n=p+\ell}^{a_t} \left( 1 + \left( \frac{s\sqrt{n}}{t\sqrt{p}} \right)^{d+1} \right) e^{-\frac{\epsilon q' (\sqrt{n} - \sqrt{p+1})^2 t}{4s}} \\ &\leq C \int_{p+1}^{\infty} e^{-\frac{\epsilon q' (\sqrt{x} - \sqrt{p+1})^2 t}{4s}} dx \\ &\quad + \left( \frac{Cs}{t\sqrt{p}} \right)^{d+1} \int_{p+1}^{\infty} x^{(d+1)/2} e^{-\frac{\epsilon q' (\sqrt{x} - \sqrt{p+1})^2 t}{4s}} dx. \end{aligned} \quad (3.44)$$

The first integral on the right-hand side has already been estimated in (3.39), for the second integral, there holds

$$\begin{aligned} \int_{p+1}^{\infty} x^{(d+1)/2} e^{-\frac{\epsilon q' (\sqrt{x} - \sqrt{p+1})^2 t}{4s}} dx &= \int_0^{\infty} (y + \sqrt{p+1})^{d+2} e^{-\frac{\epsilon q' y^2 t}{4s}} dy \\ &\leq C \int_0^{\infty} y^{d+2} e^{-\frac{\epsilon q' y^2 t}{4s}} dy + Cp^{1+\frac{d}{2}} \int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} dy \\ &\leq C \left( \frac{s}{t} \right)^{2+\frac{d}{2}} \int_0^{\infty} z^{(d+1)/2} e^{-\frac{\epsilon q' z^2}{4}} dz \\ &\quad + C \left( \frac{s}{t} \right)^{3/2} p^{1+\frac{d}{2}} \int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} dz. \end{aligned} \quad (3.45)$$

Combining (3.39), (3.44) and (3.45), we derive (3.35).

*Step 2.* Since  $\mathcal{T}_p^* \subset \Gamma_p \times [0, t]$  where  $\Gamma_p = B_{d_{p+1}}(x) \setminus B_{d_{p-1}}(x)$ ,  $(y, s) \in \mathcal{T}_p^*$  implies that  $|x - y|^2 \geq (p-1)t$ , thus  $J'_{2,\ell}{}^h$  satisfies

$$\begin{aligned} J'_{2,\ell}{}^h &\leq Ct^{\frac{1-q}{2}} \sum_{p=1}^{\infty} p^{\frac{q-1}{2}} \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-\frac{|x-y|^2}{4(t-s)}} \\ &\quad \times \sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} e^{-\frac{q\lambda_{n,j,y}^2(1-\epsilon)}{4s}} \mu_{n,j}^q(K_{n,j}) ds dy \\ &\leq Ct^{\frac{1-q}{2}} \sum_{n=\ell+1}^{a_t} \sum_{j \in \Theta_{t,n}^h} \mu_{n,j}^q(K_{n,j}) \\ &\quad \times \sum_{p=1}^{n-\ell} p^{\frac{q-1}{2}} \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-|x-y|^2/4(t-s)} e^{-\frac{q\lambda_{n,j,y}^2(1-\epsilon)}{4s}} ds dy \end{aligned} \quad (3.46)$$

and the constant  $C$  depends on  $N, q$  and  $\epsilon$ . Next we set  $q_\epsilon = (1-\epsilon)q$ . Writting

$$|y - a_{n,j}|^2 = |x - y|^2 + |x - a_{n,j}|^2 - 2\langle y - x, a_{n,j} - x \rangle \geq pt + |x - a_{n,j}|^2 - 2\langle y - x, a_{n,j} - x \rangle,$$

we get

$$\int_{\Gamma_p} e^{-\frac{q_\epsilon |y-a_{n,j}|^2}{4s}} dy = e^{-\frac{q_\epsilon |x-a_{n,j}|^2}{4s}} \int_{\sqrt{tp}}^{\sqrt{t(p+1)}} e^{-\frac{q_\epsilon r^2}{4s}} \int_{|x-y|=r} e^{2q_\epsilon \langle y-x, a_{n,j}-x \rangle / 4s} dS_r(y) dr.$$

For estimating the value of the spherical integral, we can assume that  $a_{n,j}-x = (0, \dots, 0, |a_{n,j}-x|)$ ,  $y = (y_1, \dots, y_N)$  and, using spherical coordinates with center at  $x$ , that the unit sphere has the representation  $S^{N-1} = \{(\sin \phi, \sigma, \cos \phi) \in \mathbb{R}^{N-1} \times \mathbb{R} : \sigma \in S^{N-2}, \phi \in [0, \pi]\}$ . With this representation,  $dS_r = r^{N-1} \sin^{N-2} \phi d\phi d\sigma$  and  $\langle y-x, a_{n,j}-x \rangle = |a_{n,j}-x| |y-x| \cos \phi$ . Therefore

$$\int_{|x-y|=r} e^{2q_\epsilon \frac{\langle y-x, a_{n,j}-x \rangle}{4s}} dS_r(y) = r^{N-1} |S^{N-2}| \int_0^\pi e^{2q_\epsilon \frac{|a_{n,j}-x| r \cos \phi}{4s}} \sin^{N-2} \phi d\phi.$$

By Lemma A.3

$$\begin{aligned} \int_{|x-y|=r} e^{2q_\epsilon \frac{\langle y-x, a_{n,j}-x \rangle}{4s}} dS_r(y) &\leq C \frac{r^{N-1} e^{2q_\epsilon \frac{r|a_{n,j}-x|}{4s}}}{\left(1 + \frac{r|a_{n,j}-x|}{s}\right)^{\frac{N-1}{2}}} \\ &\leq C s^{\frac{N-1}{2}} \left(\frac{r}{|a_{n,j}-x|}\right)^{\frac{N-1}{2}} e^{2q_\epsilon \frac{r|a_{n,j}-x|}{4s}}. \end{aligned} \quad (3.47)$$

Therefore

$$\int_{\Gamma_p} e^{-q_\epsilon \frac{|y-a_{n,j}|^2}{4s}} dy \leq C t^{\frac{N-1}{4}} p^{\frac{N-3}{4}} s^{\frac{N-1}{2}} \frac{e^{-q_\epsilon \frac{(|a_{n,j}-x| - \sqrt{t(p+1)})^2}{4s}}}{|a_{n,j}-x|^{\frac{N-1}{2}}}, \quad (3.48)$$

and, since  $|a_{n,j}-x| \geq \sqrt{tn}$ ,

$$\begin{aligned} \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-\frac{|x-y|^2}{4(t-s)}} e^{-q_\epsilon \frac{\lambda_{n,j,y}^2}{4s}} dy ds \\ \leq C \frac{\sqrt{tp}^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}} \int_0^t (t-s)^{-\frac{N}{2}} s^{-\frac{(q-1)(N-1)+1}{2}} e^{-\frac{pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{tn} - \sqrt{t(p+1)})^2}{4s}} ds \\ \leq C \frac{t^{\frac{1-q(N-1)}{2}} p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}} \int_0^1 (1-s)^{-\frac{N}{2}} s^{-\frac{(q-1)(N-1)+1}{2}} e^{-\frac{p}{4(1-s)}} e^{-q_\epsilon \frac{(\sqrt{n} - \sqrt{p+1})^2}{4s}} ds. \end{aligned} \quad (3.49)$$

We apply Lemma A.1, with  $A = \sqrt{p}$ ,  $B = \sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1})$ ,  $b = \frac{(q-1)(N-1)+1}{2}$ ,  $a = \frac{N}{2}$  and  $\kappa = \sqrt{q_\epsilon}(\ell-1)/8$  as in the case  $N=1$ , and noticing that, for these specific values,

$$\begin{aligned} A^{1-a} B^{1-b} (A+B)^{a+b-2} &= p^{\frac{2-N}{4}} (\sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1}))^{\frac{1-(q-1)(N-1)}{2}} \\ &\quad \times (\sqrt{p} + \sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1}))^{\frac{(q-1)(N-1)+N-3}{2}} \\ &\leq C \left(\frac{n}{p}\right)^{\frac{N}{4}-1/2} \left(\frac{\sqrt{n} - \sqrt{p}}{\sqrt{n}}\right)^{\frac{1-(q-1)(N-1)}{2}}, \end{aligned}$$

where  $C$  depends on  $N$ ,  $q$  and  $\kappa$ . Therefore

$$\begin{aligned} & \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} e^{-q_\epsilon |y-z|^2/4s} dy ds \\ & \leq C \frac{t^{(1-q(N-1))/2} p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}} \left(\frac{n}{p}\right)^{\frac{N}{4}-1/2} \left(\frac{\sqrt{n}-\sqrt{p}}{\sqrt{n}}\right)^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}} \\ & \leq C t^{\frac{1-q(N-1)}{2}} p^{-\frac{1}{4}} n^{\frac{(q-1)(N-1)-2}{4}} (\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}. \end{aligned} \quad (3.50)$$

We derive from (3.46), (3.50),

$$\begin{aligned} J'_{2,\ell} h & \leq C t^{1-\frac{Nq}{2}} \\ & \times \sum_{n=\ell+1}^{a_t} \sum_{j \in \Theta_{t,n}^h} n^{\frac{(q-1)(N-1)-2}{4}} \mu_{n,j}^q(K_{n,j}) \sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} (\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}. \end{aligned} \quad (3.51)$$

By Lemma A.2 with  $\alpha = \frac{2q-3}{4}$ ,  $\beta = \frac{1-(q-1)(N-1)}{2}$ ,  $\delta = \frac{1}{4}$  and  $\gamma = q_\epsilon$ , we obtain

$$\sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} (\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}} \leq C n^{\frac{N(q-1)+q-3}{4}} e^{-\frac{n}{4}}, \quad (3.52)$$

thus

$$J'_{2,\ell} h \leq C t^{1-\frac{Nq}{2}} \sum_{n=\ell+1}^{a_t} n^{\frac{N(q-1)}{2}-1} e^{-\frac{n}{4}} \sum_{j \in \Theta_{t,n}^h} \mu_{n,j}^q(K_{n,j}). \quad (3.53)$$

Because

$$\mu_{n,j}(K_{n,j}) = C_{2/q,q'}^{B_{n,j}}(K_{n,j}),$$

we use the rescaling procedure as in the proof of Lemma 2.23, except that the scale factor is  $\sqrt{(n+1)t}$  instead of  $\sqrt{n+1}$  so that the sets  $\tilde{T}_n$ ,  $\tilde{K}_n$ ,  $\tilde{Q}_n$  and  $\tilde{K}_n$  remains unchanged Using

again the quasi-additivity and the fact that  $J'_{2,\ell} = \sum_{h=1}^J J'_{2,\ell} h$ , we deduce

$$J_{2,\ell} \leq C' t^{-\frac{N}{2}} \sum_{n=\ell+1}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'} \left( \frac{K_n}{d_{n+1}} \right), \quad (3.54)$$

which implies (3.32).  $\square$

The proof of Theorem 3.1 follows from the previous estimates on  $J_1$  and  $J_2$ . Furthermore the following integral expression holds

**Theorem 3.8** *Assume  $q \geq q_c$ . Then there exists a positive constants  $C_2^*$ , depending on  $N, q$  and  $T$ , such that for any closed set  $F$ , there holds*

$$\underline{u}_F(x, t) \geq \frac{C_2^*}{t^{1+\frac{N}{2}}} \int_0^{\sqrt{ta_t}} e^{-\frac{s^2}{4t}} s^{N-\frac{2}{q-1}} C_{2/q,q'} \left( \frac{F}{s} \cap B_1(x) \right) s ds, \quad (3.55)$$

where  $a_t$  is the smallest integer  $j$  such that  $F \subset B_{\sqrt{jt}}(x)$ .



*Proof.* We distinguish according  $q = q_c$ , or  $q > q_c$ , and for simplicity we denote  $B_r = B_r(x)$  for the various values of  $r$ .

*Case 1:*  $q = q_c \iff N - \frac{2}{q-1} = 0$ . Because  $F_n = F \cap (B_{d_{n+1}} \setminus B_{d_n})$  there holds

$$C_{2/q,q'} \left( \frac{F_n}{d_{n+1}} \right) \geq C_{2/q,q'} \left( \frac{F}{d_{n+1}} \cap B_1 \right) - C_{2/q,q'} \left( \frac{F \cap B_{d_n}}{d_{n+1}} \right),$$

Furthermore, since  $d_{n+1} \geq d_n$ ,

$$C_{2/q,q'} \left( \frac{F \cap B_{d_n}}{d_{n+1}} \right) = C_{2/q,q'} \left( \frac{d_n}{d_{n+1}} \frac{F \cap B_{d_n}}{d_n} \right) \leq C_{2/q,q'} \left( \frac{F}{d_n} \cap B_1 \right),$$

thus

$$C_{2/q,q'} \left( \frac{F_n}{d_{n+1}} \right) \geq C_{2/q,q'} \left( \frac{F}{d_{n+1}} \cap B_1 \right) - C_{2/q,q'} \left( \frac{F}{d_n} \cap B_1 \right),$$

it follows

$$\begin{aligned} \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'} \left( \frac{F_n}{d_{n+1}} \right) &\geq \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'} \left( \frac{F}{d_{n+1}} \cap B_1 \right) - \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'} \left( \frac{F}{d_n} \cap B_1 \right) \\ &\geq \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'} \left( \frac{F}{d_{n+1}} \cap B_1 \right) - e^{-\frac{1}{4}} \sum_{n=0}^{a_t-1} e^{-\frac{n}{4}} C_{2/q,q'} \left( \frac{F}{d_{n+1}} \cap B_1 \right) \\ &\geq (1 - e^{-\frac{1}{4}}) \sum_{n=1}^{a_t-1} e^{-\frac{n}{4}} C_{2/q,q'} \left( \frac{F}{d_{n+1}} \cap B_1 \right) - e^{-\frac{1}{4}} C_{2/q,q'} \left( \frac{F}{\sqrt{t}} \cap B_1 \right). \end{aligned}$$

Since, by (2.92),

$$C_{2/q,q'} \left( \frac{F}{s'} \cap B_1 \right) \geq C_{2/q,q'} \left( \frac{F}{d_{n+1}} \cap B_1 \right) \geq C_{2/q,q'} \left( \frac{F}{s} \cap B_1 \right),$$

for any  $s' \in [d_{n+1}, d_{n+2}]$  and  $s \in [d_n, d_{n+1}]$ , there holds

$$\begin{aligned} te^{-\frac{n}{4}} C_{2/q,q'} \left( \frac{F}{d_{n+1}} \cap B_1 \right) &\geq C_{2/q,q'} \left( \frac{F}{d_{n+1}} \cap B_1 \right) \int_{d_n}^{d_{n+1}} e^{-s^2/4t} s \, ds \\ &\geq \int_{d_n}^{d_{n+1}} e^{-s^2/4t} C_{2/q,q'} \left( \frac{F}{s} \cap B_1 \right) s \, ds. \end{aligned}$$

This implies

$$W_F(x, t) \geq (1 - e^{-\frac{1}{4}}) t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} e^{-s^2/4t} C_{2/q,q'} \left( \frac{F}{s} \cap B_1 \right) s \, ds.$$

*Case 2:*  $q > q_c \iff N - \frac{2}{q-1} > 0$ . In that case it follows from Lemma 2.9 that

$$C_{2/q,q'} \left( \frac{F_n}{d_{n+1}} \right) \approx d_{n+1}^{\frac{2}{q-1}-N} C_{2/q,q'}(F_n).$$

Thus

$$W_F(x, t) \approx t^{-1-\frac{N}{2}} \sum_{n=0}^{a_t} e^{-\frac{n}{4}} C_{2/q, q'}(F_n).$$

Since

$$C_{2/q, q'}(F_n) \geq C_{2/q, q'}(F \cap B_{d_{n+1}}) - C_{2/q, q'}(F \cap B_{d_n}),$$

and again

$$\begin{aligned} t^{-\frac{N}{2}} \sum_{n=0}^{a_t} e^{-\frac{n}{4}} C_{2/q, q'}(F_n) &\geq (1 - e^{-\frac{1}{4}}) t^{-\frac{N}{2}} \sum_{n=0}^{a_t-1} e^{-\frac{n}{4}} C_{2/q, q'}(F \cap B_{d_{n+1}}) \\ &\geq (1 - e^{-\frac{1}{4}}) t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} e^{-\frac{s^2}{4t}} C_{2/q, q'}(F \cap B_s) s ds. \end{aligned}$$

Because  $C_{2/q, q'}(F \cap B_s) \approx s^{N-\frac{2}{q-1}} C_{2/q, q'}(s^{-1}F \cap B_1)$ , (3.55) follows.  $\square$

## 4 Applications

The first result of this section is the following

**Theorem 4.1** *Assume  $N \geq 1$  and  $q > 1$ . Then  $\bar{u}_K = \underline{u}_K$ .*

*Proof.* If  $1 < q < q_c$ , the result is already proved in [28]. The proof in the super-critical case is an adaptation that we recall, for the sake of completeness. By Theorem 2.24 and Theorem 3.8 there exists a positive constant  $C$ , depending on  $N$ ,  $q$  and  $T$  such that

$$\bar{u}_F(x, t) \leq C \underline{u}_F(x, t) \quad \forall (x, t) \in Q_T.$$

By convexity  $\tilde{u} = \underline{u}_F - \frac{1}{2C}(\bar{u}_F - \underline{u}_F)$  is a super-solution, which is smaller than  $\underline{u}_F$  if we assume that  $\bar{u}_F \neq \underline{u}_F$ . If we set  $\theta := 1/2 + 1/(2C)$ , then  $u_\theta = \theta \bar{u}_F$  is a subsolution. Therefore there exists a solution  $u_1$  of (1.1) in  $Q_\infty$  such that  $u_\theta \leq u_1 \leq \tilde{u} < \underline{u}_F$ . If  $\mu \in \mathfrak{M}_+^q(\mathbb{R}^N)$  satisfies  $\mu(F^c) = 0$ , then  $u_{\theta\mu}$  is the smallest solution of (1.1) which is above the subsolution  $\theta u_\mu$ . Thus  $u_{\theta\mu} \leq u_1 < \underline{u}_F$  and finally  $\underline{u}_F \leq u_1 < \underline{u}_F$ , a contradiction.  $\square$

If we combine Theorem 2.24 and Theorem 3.8 we derive the following integral approximation of the parabolic capacitary potential

**Proposition 4.2** *Assume  $q \geq q_c$ . Then there exist two positive constants  $C_1^\dagger, C_2^\dagger$ , depending only on  $N, q$  and  $T$  such that*

$$\begin{aligned} C_2^\dagger t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q, q'}\left(\frac{F}{s} \cap B_1(x)\right) s ds &\leq W_F(x, t) \\ &\leq C_1^\dagger t^{-(1+\frac{N}{2})} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q, q'}\left(\frac{F}{s} \cap B_1(x)\right) s ds \end{aligned} \quad (4.56)$$

for any  $(x, t) \in Q_T$ .

**Definition 4.3** If  $F$  is a closed subset of  $\mathbb{R}^N$ , we define the  $(2/q, q')$ -integral parabolic capacity potential  $\mathcal{W}_F$  by

$$\mathcal{W}_F(x, t) = t^{-1-\frac{N}{2}} \int_0^{D_F(x)} s^{N-\frac{2}{q-1}} e^{-s^2/4t} C_{2/q, q'} \left( \frac{F}{s} \cap B_1(x) \right) s \, ds \quad \forall (x, t) \in Q_\infty, \quad (4.57)$$

where  $D_F(x) = \max\{|x - y| : y \in F\}$ .

An easy computation shows that

$$\begin{aligned} 0 \leq \mathcal{W}_F(x, t) - t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q, q'} \left( \frac{F}{s} \cap B_1(x) \right) s \, ds \\ \leq C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-D_F^2(x)/4t}, \end{aligned} \quad (4.58)$$

and

$$\begin{aligned} 0 \leq t^{-(1+\frac{N}{2})} \int_0^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q, q'} \left( \frac{F}{s} \cap B_1(x) \right) s \, ds - \mathcal{W}_F(x, t) \\ \leq C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-\frac{D_F^2(x)}{4t}}, \end{aligned} \quad (4.59)$$

for some  $C = C(N, q) > 0$ . Furthermore

$$\mathcal{W}_F(x, t) = t^{-\frac{1}{q-1}} \int_0^{D_F(x)/\sqrt{t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4}} C_{2/q, q'} \left( \frac{F}{s\sqrt{t}} \cap B_1(x) \right) s \, ds. \quad (4.60)$$

The following result gives a sufficient condition in order that  $\bar{u}_F$  does not have a strong blow-up at a point  $x$ .

**Proposition 4.4** Assume  $q \geq q_c$  and  $F$  is a closed subset of  $\mathbb{R}^N$ . If there exists  $\gamma \in [0, \infty)$  such that

$$\lim_{\tau \rightarrow 0} C_{2/q, q'} \left( \frac{F}{\tau} \cap B_1(x) \right) = \gamma, \quad (4.61)$$

then

$$\lim_{t \rightarrow 0} t^{\frac{1}{q-1}} \bar{u}_F(x, t) = C\gamma, \quad (4.62)$$

for some  $C = C(N, q) > 0$ .

*Proof.* Clearly, condition (4.61) implies

$$\lim_{t \rightarrow 0} C_{2/q, q'} \left( \frac{F}{\sqrt{ts}} \cap B_1(x) \right) = \gamma$$

for any  $s > 0$ . Then (4.62) follows by Lebesgue's theorem. Notice also that the set of  $\gamma$  is bounded from above by a constant depending on  $N$  and  $q$ .  $\square$

In the next result we give a condition in order that the solution remains bounded at a point  $x$ . The proof is similar to the previous one.

**Proposition 4.5** Assume  $q \geq q_c$  and  $F$  is a closed subset of  $\mathbb{R}^N$ . If

$$\limsup_{\tau \rightarrow 0} \tau^{-\frac{2}{q-1}} C_{2/q, q'} \left( \frac{F}{\tau} \cap B_1(x) \right) < \infty, \quad (4.63)$$

then  $\bar{u}_F(x, t)$  remains bounded when  $t \rightarrow 0$ .

*Remark.* If we assume that  $f$  is a convex function on  $\mathbb{R}^+$  satisfying

$$c_2 r^q \leq f(r) \leq c_1 r^q \quad \forall r \geq 0 \quad (4.64)$$

for some  $0 < c_2 \leq c_1$  we can construct in the same way as for (1.1) the solutions  $\underline{u}_F$  and  $\bar{u}_F$  for equation

$$\partial_t u - \Delta u + f(u) = 0 \quad \text{in } Q_T. \quad (4.65)$$

The bilateral estimate estimate (1.19) is still valid (up to change of the  $C_i$ ). Since only convexity of  $f$  is used in the proof of Theorem 4.1, there still holds  $\underline{u}_F = \bar{u}_F$ . Similar extensions of Proposition 4.4 and Proposition 4.5 are also clear.

## A Appendix

The next estimate is crucial in our study of semilinear parabolic equations.

**Lemma A.1** Let  $a$  and  $b$  be two real numbers,  $a > 0$  and  $\kappa > 0$ . Then there exists a constant  $C = C(a, b, \kappa) > 0$  such that for any  $A > 0$ ,  $B > \kappa/A$  there holds

$$\int_0^1 (1-x)^{-a} x^{-b} e^{-A^2/4(1-x)} e^{-B^2/4x} dx \leq C e^{-(A+B)^2/4} A^{1-a} B^{1-b} (A+B)^{a+b-2}. \quad (A.1)$$

*Proof.* We first notice that

$$\max\{e^{-A^2/4(1-x)} e^{-B^2/4x} : 0 \leq x \leq 1\} = e^{-(A+B)^2/4}, \quad (A.2)$$

and it is achieved for  $x_0 = B/(A+B)$ . Set  $\Phi(x) = (1-x)^{-a} x^{-b} e^{-A^2/4(1-x)} e^{-B^2/4x}$ , thus

$$\int_0^1 \Phi(x) dx = \int_0^{x_0} \Phi(x) dx + \int_{x_0}^1 \Phi(x) dx = I_{a,b} + J_{a,b}.$$

Put

$$u = \frac{A^2}{4(1-x)} + \frac{B^2}{4x}, \quad (A.3)$$

then

$$4ux^2 - (4u + B^2 - A^2)x + B^2 = 0. \quad (A.4)$$

If  $0 < x < x_0$  this equation admits the solution

$$x = x(u) = \frac{1}{8u} \left( 4u + B^2 - A^2 - \sqrt{16u^2 - 8u(A^2 + B^2) + (A^2 - B^2)^2} \right)$$

$$\int_0^{x_0} (1-x)^{-a} x^{-b} e^{-A^2/4(1-x)-B^2/4x} dx = - \int_{(A+B)^2/4}^{\infty} (1-x(u))^{-a} x(u)^{-b} e^{-u} x'(u) du$$

Putting  $x' = x'(u)$  and differentiating (A.4),

$$4x^2 + 8uxx' - (4u + B^2 - A^2)x' - 4x = 0 \implies -x' = \frac{4x(1-x)}{4u + B^2 - A^2 - 8ux}.$$

Thus

$$\int_0^{x_0} \Phi(x) dx = 4 \int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1} x(u)^{-b+1} e^{-u} du}{4u + B^2 - A^2 - 8ux(u)}. \quad (\text{A.5})$$

Using the explicit value of the root  $x(u)$ , we finally get

$$\int_0^{x_0} \Phi(x) dx = 4 \int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1} x(u)^{-b+1} e^{-u} du}{\sqrt{16u^2 - 8u(A^2 + B^2) + (A^2 - B^2)^2}}, \quad (\text{A.6})$$

and the factorization below holds

$$16u^2 - 8u(A^2 + B^2) + (A^2 - B^2)^2 = 16(u - (A+B)^2/4)(u - (A-B)^2/4).$$

We set  $u = v + (A+B)^2/4$  and obtain

$$x(u) = \frac{v + (AB + B^2)/2 - \sqrt{v(v+AB)}}{2(v + (A+B)^2/4)},$$

and

$$1 - x(u) = \frac{v + (A^2 + AB)/2 + \sqrt{v(v+AB)}}{2(v + (A+B)^2/4)}.$$

We introduce the relation  $\approx$  linking two positive quantities depending on  $A$  and  $B$ . It means that the two sided-inequalities up to multiplicative constants independent of  $A$  and  $B$ . Therefore

$$\begin{aligned} \int_0^{x_0} \Phi(x) dx &= 2^{a-b-4} e^{-(A+B)^2/4} \int_0^{\infty} \tilde{\Phi}(v) dv \quad \text{where} \\ \tilde{\Phi}(v) &= \frac{\left(v + (AB + B^2)/2 - \sqrt{v(v+AB)}\right)^{1-b} \left(v + (A^2 + AB)/2 + \sqrt{v(v+AB)}\right)^{1-a}}{(v + (A+B)^2/4)^{2-a-b} \sqrt{v(v+AB)}} e^{-v} dv. \end{aligned} \quad (\text{A.7})$$

*Case 1:  $a \geq 1, b \geq 1$ . First*

$$\frac{(v + (A+B)^2/4)^{a+b-2}}{\sqrt{v(v+AB)}} \leq \frac{(v + (A+B)^2/4)^{a+b-2}}{\sqrt{v(v+\kappa)}} \approx \frac{(v + (A+B)^2)^{a+b-2}}{\sqrt{v(v+\kappa)}} \quad (\text{A.8})$$

since  $a+b-2 \geq 0$  and  $AB \geq \kappa$ . Next

$$\left(v + (A^2 + AB)/2 + \sqrt{v(v+AB)}\right)^{1-a} \approx (v + A(A+B))^{1-a}. \quad (\text{A.9})$$

Furthermore

$$\begin{aligned} v + (AB + B^2)/2 - \sqrt{v(v + AB)} &= B^2 \frac{v + (A + B)^2/4}{v + B(A + B)/2 + \sqrt{v(v + AB)}} \\ &\approx B^2 \frac{v + (A + B)^2}{v + B(A + B)}. \end{aligned} \quad (\text{A.10})$$

Then

$$\left( v + (AB + B^2)/2 - \sqrt{v(v + AB)} \right)^{1-b} \approx B^{2-2b} \left( \frac{v + B(A + B)}{v + (A + B)^2} \right)^{b-1} \quad (\text{A.11})$$

It follows

$$\begin{aligned} \tilde{\Phi}(v) &\leq CB^{2-2b} \left( \frac{v + (A + B)^2}{v + A(A + B)} \right)^{a-1} \frac{(v + B(A + B))^{b-1}}{\sqrt{v(v + \kappa)}} \\ &\leq CB^{2-2b} \left( \frac{v + (A + B)^2}{v + A(A + B)} \right)^{a-1} \frac{v^{b-1} + (B^2 + AB)^{b-1}}{\sqrt{v(v + \kappa)}} \end{aligned} \quad (\text{A.12})$$

where  $C$  depends on  $a$ ,  $b$  and  $\kappa$ . The function  $v \mapsto (v + (A + B)^2)/(v + A(A + B))$  is decreasing on  $(0, \infty)$ . If we set

$$C_1 = \int_0^\infty \frac{v^{b-1} e^{-v} dv}{\sqrt{v(v + \kappa)}} \quad \text{and} \quad C_2 = \int_0^\infty \frac{e^{-v} dv}{\sqrt{v(v + \kappa)}}$$

then

$$C_1 \leq K(B^2 + AB)^{b-1} C_2$$

with  $K = C_1 \kappa^{1-b}/C_2$ . Therefore

$$\int_0^{x_0} \Phi(x) dx \leq C e^{-(A+B)^2/4} B^{1-b} A^{1-a} (A + B)^{a+b-2}. \quad (\text{A.13})$$

The estimate of  $J_{a,b}$  is obtained by exchanging  $(A, a)$  with  $(B, b)$  and replacing  $x$  by  $1 - x$ . *Mutadis mutandis*, this yields directly to the same expression as in A.13 and finally

$$\int_0^1 \Phi(x) dx \leq C e^{-(A+B)^2/4} A^{1-a} B^{1-b} (A + B)^{a+b-2}. \quad (\text{A.14})$$

*Case 2:*  $a \geq 1$ ,  $b < 1$ . Estimates (A.7), (A.8), (A.9), (A.10) and (A.11) are valid. Because  $v \mapsto (v + B(A + B))^{b-1}$  is decreasing, (A.12) has to be replaced by

$$\tilde{\Phi}(v) \leq CB^{2-2b} \left( \frac{v + (A + B)^2}{v + A(A + B)} \right)^{a-1} \frac{(AB + B^2)^{b-1}}{\sqrt{v(v + \kappa)}}. \quad (\text{A.15})$$

This implies (A.13) directly. The estimate of  $J_{a,b}$  is performed by the change of variable  $x \mapsto 1 - x$ . If  $x_1 = 1 - x_0$ , there holds

$$J_{a,b} = \int_0^{x_1} x^{-a} (1 - x)^{-b} e^{-A^2/4x} e^{-B^2/4(1-x)} dx = \int_0^{x_1} \Psi(x) dx.$$

Then

$$\int_0^{x_1} \Psi(x) dx = 2^{b-a-4} e^{-(A+B)^2/4} \int_0^{x_1} \tilde{\Psi}(v) dv \quad \text{where}$$

$$\tilde{\Psi}(v) = \frac{\left(v + (AB + A^2)/2 - \sqrt{v(v+AB)}\right)^{1-a} \left(v + (B^2 + AB)/2 + \sqrt{v(v+AB)}\right)^{1-b}}{(v + (A+B)^2/4)^{2-a-b} \sqrt{v(v+AB)}} e^{-v} dv. \quad (\text{A.16})$$

Equivalence (A.8) is unchanged; (A.9) is replaced by

$$\left(v + (B^2 + AB)/2 + \sqrt{v(v+AB)}\right)^{1-b} \approx (v + B(A+B))^{1-b}, \quad (\text{A.17})$$

(A.10) by

$$v + (AB + A^2)/2 - \sqrt{v(v+AB)} \approx A^2 \frac{v + (A+B)^2}{v + A(A+B)}, \quad (\text{A.18})$$

and (A.11) by

$$\left(v + (AB + A^2)/2 - \sqrt{v(v+AB)}\right)^{1-a} \approx A^{2-2a} \left(\frac{v + A(A+B)}{v + (A+B)^2}\right)^{a-1}. \quad (\text{A.19})$$

Because  $a > 1$ , (A.12) turns into

$$\begin{aligned} \tilde{\Psi}(v) &\leq C A^{2-2b} (v + (A+B)^2)^{b-1} \frac{(v + A^2 + AB)^{a-1} (v + B^2 + AB)^{1-b}}{\sqrt{v(v+\kappa)}} \\ &\leq C e^{-(A+B)^2/4} A^{2-2b} (A+B)^{2b-2} \\ &\quad \times \frac{v^{a-b} + (A^2 + AB)^{a-1} v^{1-b} + (B^2 + AB)^{1-b} v^{a-1} + A^{a-1} B^{1-b} (A+B)^{a-b}}{\sqrt{v(v+\kappa)}}. \end{aligned} \quad (\text{A.20})$$

Because  $AB \geq \kappa$ , there exists a positive constant  $C$ , depending on  $\kappa$ , such that

$$\begin{aligned} \int_0^\infty \frac{v^{a-b} + (A^2 + AB)^{a-1} v^{1-b} + (B^2 + AB)^{1-b} v^{a-1}}{\sqrt{v(v+\kappa)}} e^{-v} dv \\ \leq C A^{a-1} B^{1-b} (A+B)^{a-b} \int_0^\infty \frac{e^{-v} dv}{\sqrt{v(v+\kappa)}}. \end{aligned} \quad (\text{A.21})$$

Combining (A.20) and (A.21) yields to

$$\int_0^{x_1} \Psi(x) dx \leq C e^{-(A+B)^2/4} A^{1-a} B^{1-b} (A+B)^{a+b-2}. \quad (\text{A.22})$$

This, again, implies that (A.1) holds.

*Case 3:*  $\max\{a, b\} < 1$ . Inequalities (A.7)-(A.11) hold, but (A.12) has to be replaced by

$$\begin{aligned} \tilde{\Phi}(v) &\leq C B^{2-2b} \left(\frac{v + (A+B)^2}{v + A(A+B)}\right)^{a-1} \frac{(v + B^2 + AB)^{b-1}}{\sqrt{v(v+\kappa)}} \\ &\leq C B^{1-b} (A+B)^{2a+b-3} \frac{v^{1-a} + (A^2 + AB)^{1-a}}{\sqrt{v(v+\kappa)}} \end{aligned} \quad (\text{A.23})$$

Noticing that

$$\int_0^\infty \frac{v^{1-a} e^{-v} dv}{\sqrt{v(v+\kappa)}} \leq C (A^2 + AB)^{1-a} \int_0^\infty \frac{e^{-v} dv}{\sqrt{v(v+\kappa)}},$$

it follows that (A.13) holds. Finally (A.14) holds by exchanging  $(A, a)$  and  $(B, b)$ .  $\square$

**Lemma A.2** . Let  $\alpha, \beta, \gamma, \delta$  be real numbers and  $\ell$  an integer. We assume  $\gamma > 1, \delta > 0$  and  $\ell \geq 2$ . Then there exists a positive constant  $C$  such that, for any integer  $n > \ell$

$$\sum_{p=1}^{n-\ell} p^\alpha (\sqrt{n} - \sqrt{p})^\beta e^{-\delta(\sqrt{p} + \sqrt{\gamma}(\sqrt{n} - \sqrt{p+1}))^2} \leq C n^{\alpha-\beta/2} e^{-\delta n}. \quad (\text{A.24})$$

*Proof.* The function  $x \mapsto (\sqrt{x} + \sqrt{\gamma}(\sqrt{n} - \sqrt{x+1}))^2$  is decreasing on  $[(\gamma-1)^{-1}, \infty)$ . Furthermore there exists  $C > 0$  depending on  $\ell, \alpha$  and  $\beta$  such that  $p^\alpha (\sqrt{n} - \sqrt{p})^\beta \leq C x^\alpha (\sqrt{n} - \sqrt{x+1})^\beta$  for  $x \in [p, p+1]$ . If we denote by  $p_0$  the smallest integer larger than  $(\gamma-1)^{-1}$ , we derive

$$\begin{aligned} S &= \sum_{p=1}^{n-\ell} p^\alpha (\sqrt{n} - \sqrt{p})^\beta e^{-(\sqrt{p} + \sqrt{\gamma}(\sqrt{n} - \sqrt{p+1}))^2/4} = \sum_{p=1}^{p_0-1} + \sum_{p_0}^{n-\ell} p^\alpha (\sqrt{n} - \sqrt{p})^\beta e^{-\delta(\sqrt{p} + \sqrt{\gamma}(\sqrt{n} - \sqrt{p+1}))^2} \\ &\leq \sum_{p=1}^{p_0-1} p^\alpha (\sqrt{n} - \sqrt{p})^\beta e^{-\delta(\sqrt{p} + \sqrt{\gamma}(\sqrt{n} - \sqrt{p+1}))^2} \\ &\quad + C \int_{p_0}^{n+1-\ell} x^\alpha (\sqrt{n} - \sqrt{x})^\beta e^{-\delta(\sqrt{x} + \sqrt{\gamma}(\sqrt{n} - \sqrt{x+1}))^2} dx, \end{aligned}$$

(notice that  $\sqrt{n} - \sqrt{x} \approx \sqrt{n} - \sqrt{x+1}$  for  $x \leq n - \ell$ ). Clearly

$$\sum_{p=1}^{p_0-1} p^\alpha (\sqrt{n} - \sqrt{p})^\beta e^{-\delta(\sqrt{p} + \sqrt{\gamma}(\sqrt{n} - \sqrt{p+1}))^2} \leq C_0 n^\alpha (\sqrt{n} - \sqrt{n-\ell})^\beta e^{-\delta n} \quad (\text{A.25})$$

for some  $C_0$  independent of  $n$ . We set  $y = y(x) = \sqrt{x+1} - \sqrt{x}/\sqrt{\gamma}$ . Obviously

$$y'(x) = \frac{1}{2} \left( \frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{\gamma}\sqrt{x}} \right) \quad \forall x \geq p_0,$$

and there exists  $\epsilon = \epsilon(\delta, \gamma) > 0$  such that  $\sqrt{2}\sqrt{x} \geq y(x) \geq \epsilon\sqrt{x}$  and  $y'(x) \geq \epsilon/\sqrt{x}$ . Furthermore

$$\begin{aligned} \sqrt{x} &= \frac{\sqrt{\gamma} \left( y + \sqrt{\gamma y^2 + 1 - \gamma} \right)}{\gamma - 1}, \\ \sqrt{n} - \sqrt{x} &= \frac{\sqrt{n}(\gamma - 1) - \sqrt{\gamma}y - \sqrt{\gamma}\sqrt{\gamma y^2 + 1 - \gamma}}{\gamma - 1} \\ &= \frac{n(\gamma - 1) + \gamma - 2y\sqrt{\gamma n} - \gamma y^2}{\sqrt{n}(\gamma - 1) - \sqrt{\gamma}y + \sqrt{\gamma}\sqrt{\gamma y^2 + 1 - \gamma}} \\ &\approx \frac{n(\gamma - 1) + \gamma - 2y\sqrt{\gamma n} - \gamma y^2}{\sqrt{n}} \end{aligned}$$



since  $y(x) \leq \sqrt{n}$ . Furthermore

$$\begin{aligned} n(\gamma - 1) + \gamma - 2y\sqrt{\gamma n} - \gamma y^2 &= \gamma(\sqrt{n+1} + \sqrt{n}/\sqrt{\gamma} + y)(\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y) \\ &\approx \sqrt{n}(\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y), \end{aligned}$$

because  $y$  ranges between  $\sqrt{n+2-\ell} - \sqrt{n+1-\ell}\sqrt{\gamma} \approx \sqrt{n}$  and  $\sqrt{p_0+1} - \sqrt{p_0}\sqrt{\gamma}$ . Thus

$$(\sqrt{n} - \sqrt{x})^\beta \approx (\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y)^\beta.$$

This implies

$$\begin{aligned} &\int_{p_0}^{n+1-\ell} x^\alpha (\sqrt{n} - \sqrt{x})^\beta e^{-\delta(\sqrt{x} + \gamma(\sqrt{n} - \sqrt{x+1}))^2} dx \\ &\leq C \int_{y(p_0)}^{y(n+1-\ell)} y^{2\alpha+1} (\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y)^\beta e^{-\gamma\delta(\sqrt{n}-y)^2} dy \\ &\leq C n^{\alpha+\beta/2+1} \int_{1-y(n+1-\ell)/\sqrt{n}}^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz. \end{aligned} \tag{A.26}$$

Moreover

$$\begin{aligned} 1 - \frac{y(p_0)}{\sqrt{n}} &= 1 - \frac{1}{\sqrt{n}} \left( \sqrt{p_0+1} - \frac{\sqrt{p_0}}{\sqrt{\gamma}} \right), \\ 1 - \frac{y(n-\ell+1)}{\sqrt{n}} &= 1 - \frac{\sqrt{n-\ell+2}}{\sqrt{n}} + \frac{\sqrt{n-\ell+1}}{\sqrt{n}\gamma} \\ &= \frac{1}{\sqrt{\gamma}} \left( 1 + \frac{\sqrt{\gamma}(\ell-2) - \ell + 1}{2n} + \frac{\sqrt{\gamma}(\ell-2)^2 - (\ell-1)^2}{8n^2} \right) + O(n^{-3}). \end{aligned} \tag{A.27}$$

Let  $\theta$  fixed such that  $1 - \frac{y(n-\ell+1)}{\sqrt{n}} < \theta < 1 - \frac{y(p_0)}{\sqrt{n}}$  for any  $n > p_0$ . Then

$$\begin{aligned} \int_{\theta}^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz &\leq C_\theta \int_{\theta}^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} e^{-\gamma\delta n z^2} dz \\ &\leq C_\theta e^{-\gamma\delta n \theta^2} \int_{\theta}^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} dz \\ &\leq C e^{-\gamma\delta n \theta^2} \max\{1, n^{-\alpha-1/2}\}. \end{aligned}$$

Because  $\gamma\theta^2 > 1$  we derive

$$\int_{\theta}^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz \leq C n^{-\beta} e^{-\delta n}, \tag{A.28}$$

for some constant  $C > 0$ . On the other hand

$$\begin{aligned} \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (1-z)^{2\alpha+1} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz \\ \leq C'_\theta \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z + \sqrt{1+1/n} - 1 - 1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz. \end{aligned}$$

The minimum of  $z \mapsto (z + \sqrt{1 + 1/n} - 1 - 1/\sqrt{\gamma})^\beta$  is achieved at  $1 - y(n + 1 - \ell)$  with value

$$\frac{\sqrt{\gamma}(\ell + 1) + 1 - \ell}{2n\sqrt{\gamma}} + O(n^{-2}),$$

and the maximum of the exponential term is achieved at the same point with value

$$e^{-n\delta + ((\ell - 2)\sqrt{\gamma} + 1 - \ell)/2}(1 + o(1)) = C_\gamma e^{-n\delta}(1 + o(1)).$$

We denote

$$z_{\gamma,n} = 1 + 1/\sqrt{\gamma} - \sqrt{1 + 1/n} \quad \text{and} \quad I_\beta = \int_{1-y(n+1-\ell)/\sqrt{n}}^\theta (z - z_{\gamma,n})^\beta e^{-\gamma\delta n z^2} dz.$$

Since  $1 - y(n + 1 - \ell) \geq 1/\sqrt{2\gamma}$  for  $n$  large enough,

$$\begin{aligned} I_\beta &\leq \sqrt{2\gamma} \int_{1-y(n+1-\ell)/\sqrt{n}}^\theta (z - z_{\gamma,n})^\beta z e^{-\gamma\delta n z^2} dz \\ &\leq \frac{-\sqrt{2\gamma}}{2n\gamma\delta} \left[ (z - z_{\gamma,n})^\beta e^{-\gamma\delta n z^2} \right]_{1-y(n+1-\ell)/\sqrt{n}}^\theta + \frac{\beta\sqrt{2\gamma}}{2n\gamma\delta} \int_{1-y(n+1-\ell)/\sqrt{n}}^\theta (z - z_{\gamma,n})^{\beta-1} z e^{-\gamma\delta n z^2} dz \end{aligned}$$

But  $1 - y(n + 1 - \ell)/\sqrt{n} - z_{\gamma,n} = (\ell - 1)(1 - 1/\sqrt{\gamma})/2n$ , therefore

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n} + \beta C'_1 n^{-1} I_{\beta-1}. \quad (\text{A.29})$$

If  $\beta \leq 0$ , we derive

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n},$$

which inequality, combined with (A.26) and (A.28), yields to (A.24). If  $\beta > 0$ , we iterate and get

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n} + C'_1 n^{-1} (C_1 n^{-\beta} e^{-\delta n} + (\beta - 1) C'_1 n^{-1} I_{\beta-2})$$

If  $\beta - 1 \leq 0$  we derive

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n} + C_1 C'_1 n^{-1-\beta} e^{-\delta n} = C_2 n^{-\beta-1} e^{-\delta n},$$

which again yields to (A.24). If  $\beta - 1 > 0$ , we continue up we find a positive integer  $k$  such that  $\beta - k \leq 0$ , which again yields to

$$I_\beta \leq C_k n^{-\beta-1} e^{-\delta n}$$

and to (A.24). □

The next estimate is fundamental in deriving the  $N$ -dimensional estimate.

**Lemma A.3** *For any integer  $N \geq 2$  there exists a constant  $c_N > 0$  such that*

$$\int_0^\pi e^{m \cos \theta} \sin^{N-2} \theta d\theta \leq c_N \frac{e^m}{(1 + m)^{(N-1)/2}} \quad \forall m > 0. \quad (\text{A.30})$$

*Proof.* Put  $\mathcal{I}_N(m) = \int_0^\pi e^{m \cos \theta} \sin^{N-2} \theta d\theta$ . Then  $\mathcal{I}_2'(m) = \int_0^\pi e^{m \cos \theta} \cos \theta d\theta$  and

$$\begin{aligned}\mathcal{I}_2''(m) &= \int_0^\pi e^{m \cos \theta} \cos^2 \theta d\theta = \mathcal{I}_2(m) - \int_0^\pi e^{m \cos \theta} \sin^2 \theta d\theta \\ &= \mathcal{I}_2(m) - \frac{1}{m} \int_0^\pi e^{m \cos \theta} \cos \theta d\theta \\ &= \mathcal{I}_2(m) - \frac{1}{m} \mathcal{I}_2'(m).\end{aligned}$$

Thus  $\mathcal{I}_2$  satisfies a Bessel equation of order 0. Since  $\mathcal{I}_2(0) = \pi$  and  $\mathcal{I}_2'(0) = 0$ ,  $\pi^{-1}\mathcal{I}_2$  is the modified Bessel function of index 0 (usually denoted by  $I_0$ ) the asymptotic behaviour of which is well known, thus (A.30) holds. If  $N = 3$

$$\mathcal{I}_3(m) = \int_0^\pi e^{m \cos \theta} \sin \theta d\theta = \left[ \frac{-e^{m \cos \theta}}{m} \right]_0^\pi = \frac{2 \sinh m}{m}.$$

For  $N > 3$  arbitrary

$$\mathcal{I}_N(m) = \int_0^\pi \frac{-1}{m} \frac{d}{d\theta} (e^{m \cos \theta}) \sin^{N-3} \theta d\theta = \frac{N-3}{m} \int_0^\pi e^{m \cos \theta} \cos \theta \sin^{N-4} \theta d\theta. \quad (\text{A.31})$$

Therefore,

$$\mathcal{I}_4(m) = \frac{1}{m} \int_0^\pi e^{m \cos \theta} \cos \theta d\theta = \mathcal{I}_2'(m),$$

and, again (A.30) holds since  $\mathcal{I}_0'(m)$  has the same behaviour as  $I_0(m)$  at infinity. For  $N \geq 5$

$$\mathcal{I}_N(m) = \frac{3-N}{m^2} \left[ e^{m \cos \theta} \cos \theta \sin^{N-5} \theta \right]_0^\pi + \frac{N-3}{m^2} \int_0^\pi e^{m \cos \theta} \frac{d}{d\theta} (\cos \theta \sin^{N-5} \theta) d\theta.$$

Differentiating  $\cos \theta \sin^{N-5} \theta$  and using (A.31), we obtain

$$\mathcal{I}_5(m) = \frac{4 \sinh m}{m^2} - \frac{4 \sinh m}{m^3},$$

while

$$\mathcal{I}_N(m) = \frac{(N-3)(N-5)}{m^2} (\mathcal{I}_{N-4}(m) - \mathcal{I}_{N-2}(m)), \quad (\text{A.32})$$

for  $N \geq 6$ . Since the estimate (A.30) for  $\mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$  and  $\mathcal{I}_5$  has already been obtained, a straightforward induction yields to the general result.  $\square$

*Remark.* Although it does not has any importance for our use, it must be noticed that  $\mathcal{I}_N$  can be expressed either with hyperbolic functions if  $N$  is odd, or with Bessel functions if  $N$  is even.

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